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On the Width of Handles in Two-dimensional Quantum Gravity

Thordur Jonsson[1]

Raunvisindastofnun Haskolans, University of Iceland
Dunhaga 3, 107 Reykjavik
Iceland

Abstract. We discuss the average length $\bar{l}$ of the shortest non-contractible loop on surfaces in the two-dimensional pure quantum gravity ensemble. The value of γ_{str} and the explicit form of the continuum loop functions indicate that $\bar{l}$ diverges at the critical point. Scaling arguments suggest that the critical exponent of $\bar{l}$ is $\frac{1}{2}$. We show that this value of the critical exponent is also obtained for branched polymers with loops where the calculation is straightforward.

[1] e-mail: thjons@raunvis.hi.is

In spite of the fact that one can calculate the continuum loop functions of two-dimensional quantum gravity more or less explicitly [1] we do not have a very good understanding of what the generic surfaces contributing to the loop functions really look like. In some sense these surfaces are "thick" since they are not like trees or branched polymers and the "minimal baby universe" (minbu) picture of [2] has provided a valuable insight and some quantitative understanding.

In this note we discuss the beahavior of a very simple quantity which can be regarded as a measure of how far surfaces are from being branched polymers. This is the length l of the shortest non-contractible loop so we have in mind surfaces of genus $g > 0$. We consider randomly triangulated surfaces and loops which consist entirely of links on the surface. Each link is defined to have length 1 so the length of a loop is the number of links it contains assuming that each link is traversed only once. It is clear that every homotopy class of loops on a triangulated surface contains a link loop of shortest length. Let $\mathcal{S}$ denote the class of surfaces under consideration, let $\mathcal{S}_A$ be the class of surfaces of area (=number of triangles) A and let $|S|$ denote the area of a surface S. The length of the shortest noncontractible loop on S will be denoted $l(S)$. We denote the average value of l in the canonical ensemble by $\bar{l}_A$ and in the grand canonical ensemble by $\bar{l}(\mu)$. These averages are defined as

$$\bar{l}_A = N(A)^{-1} \sum_{S \in \mathcal{S}_A} l(S) \tag{1}$$

and

$$\bar{l}(\mu) = Z(\mu)^{-1} \sum_{S \in \mathcal{S}} l(S)\, e^{-\mu|S|} \tag{2}$$

where $N(A) = \#\mathcal{S}_A$ and

$$Z(\mu) = \sum_{S \in \mathcal{S}} e^{-\mu|S|} \tag{3}$$

is the partition function in the ensemble under study.

There are several reasons why it is natural to expect the average $\bar{l}(\mu)$ to be large as $\mu \to \mu_0$. First recall that the number $N_g(A)$ of closed surfaces of genus g made up of A triangles, one of which is marked, grows for large A as

$$N_g(A) \sim A^{\gamma(g)-2} e^{\mu_0 A} \tag{4}$$

where $\gamma(g) = -\frac{1}{2} + \frac{5}{2} g$ and μ_0 is independent of g. If the typical handle on a genus g surface were thin so it could be created by identifying two small regions on a surface of genus $g-1$ then we would expect $\gamma(g)$ to grow like $2g$ since the entropy associated with choosing a small group of triangles is A. For convenience we shall assume from now on that the surfaces we discuss have a marked triangle but this is not essential.

Additional evidence for the divergence of $\bar{l}(\mu)$ comes from the explicit calculation of the continuum loop functions of 2d quantum gravity [3, 1]. If the handles were thin compared to the diverging lengthscale in the continuum limit we would expect the loop functions in different genera to be related by simple overall entropic factors but this is not the case.

Perhaps the strongest argument for the macroscopic nature of handles in 2d quantum gravity is obtained by adopting the minbu picture of [2] to estimate the width of handles. The minbu picture is in good accord with all numerical and analytical facts about quantum gravity. The estimate we are interested in is in fact implicit in [2]. Let us consider surfaces of genus 1 and area A. The number of these is given asymptotically by

$$N_1(A) \sim e^{\mu_0 A}. \tag{5}$$

Let $N_1(A;l)$ denote the number of genus 0 surfaces of area A with two boundary components of length l with the property that the boundaries cannot be deformed to shorter boundaries on the surface. It follows that if we identify the two boundary components link by link we create a closed genus 1 surface with a noncontractible loop of length l which is not homotopic to a shorter loop. The authors of [2] argue that

$$N_1(A;l) \sim N_0(A)A^2 l^{-1} \tag{6}$$

for $l \leq \sqrt{A}$ while $N_1(A;l)$ is zero for $l > \sqrt{A}$. We can generically identify the two boundary loops of surfaces contributing to $N_1(A,l,l)$ in l different ways. Any surface of genus 1 (with a marked triangle) can be constructed uniquely in this way for some l. It follows that

$$\begin{aligned} N_1(A) &\sim \sum_{l=2}^{\sqrt{A}} l N_1(A;l) \\ &\sim e^{\mu_0 A} \end{aligned} \tag{7}$$

so the picture is consistent. Within the approximation embodied in (6) it follows that the probability that the shortest noncontractible loop on a torus of area A has length l is given by

$$p_A(l) = \begin{cases} A^{-1/2}, & 2 \leq l \leq \sqrt{A} \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

We see that all kinematically possible lengths are equally probable. It follows that the canonical average of l is given by

$$\bar{l}_A \sim \sum_{l=2}^{\sqrt{A}} \frac{l}{\sqrt{A}} \sim \sqrt{A}. \tag{9}$$

The corresponding grand canonical average is given by

$$\begin{aligned} \bar{l}(\mu) &= Z(\mu)^{-1} \sum_{A,l} N_1(A) p_A(l) l \, e^{-\mu A} \\ &\sim (\mu - \mu_0)^{-\frac{1}{2}}. \end{aligned} \tag{10}$$

Naively one might have expected $\bar{l}(\mu) \sim (\mu - \mu_0)^{-1/4}$ since the divergent correlation length defined by the two point function of pure gravity has critical exponent 1/4 [4]. We will now show that there is in fact no conflict between this result and (10).

The homotopy group of a torus is generated by the equivalence classes of two elementary loops l_a and l_b with the single relation $l_a l_b l_a^{-1} l_b^{-1} = 1$. Let us adopt the convention that l_a is the shortest noncontractible loop on the torus under consideration and we can take l_b to be the shortest noncontractible loop which is not homotopic to any power of l_a. In a surface which is not thin one would expect the lengths of l_a and l_b (denoted $|l_a|$ and $|l_b|$) to be comparable while in a branched polymer like surface $|l_a|$ is of order 1 while $|l_b|$ might be large. Let us denote the grand canonical expectation value of $|l_b|$ by $\bar{l}'(\mu)$. We will argue that $\bar{l}'(\mu)$ diverges with an exponent $\frac{1}{2}$ as $\mu \to \mu_0$ so there is a symmetry between the divergence of the lengths of a-loops and b-loops in pure quantum gravity. In the branched polymer phase $\bar{l}$ is bounded while $\bar{l}'$ diverges as we shall see later.

We now use the pure gravity two point function $G_\mu(r)$ to estimate $\bar{l}'$. The function $G_\mu(r)$ is defined as the partition function for all surfaces with two marked triangles separated by a distance r. The asymptotic behaviour of $G_\mu(r)$ is given by

$$G_\mu(r) \sim r^{-3} \quad \text{for } 1 \ll r \ll \xi(\mu) \tag{11}$$

and

$$G_\mu(r) \sim e^{-r/\xi} \quad \text{for } \xi(\mu) \ll r \tag{12}$$

where $\xi(\mu) \sim (\mu - \mu_0)^{-1/4}$ as $\mu \to \mu_0$ [4]. Identifying the boundaries of two triangles a distance r apart on a planar surface creates an elementary noncontractible loop of length r. For large r this loop must be l_b so we expect

$$\begin{aligned} \bar{l}'(\mu) &\sim \sum_r r G_\mu(r) \\ &\sim \xi^{-2}(\mu) \\ &\sim (\mu - \mu_0)^{-\frac{1}{2}}. \end{aligned} \tag{13}$$

We find this a suggestive argument for $\bar{l}(\mu) \sim \bar{l}'(\mu)$ in the pure gravity phase while the collapse to branched polymers for central charge $c > 1$ is accompanied by a breakdown of the symmetry between a-loops and b-loops.

Let us now turn to the calculation of the expectation value of the length of the shortest loop in a pure branched polymer ensemble. We consider planar polymers (i.e. polymers that are embedded in a plane) of genus 1 which correspond to graphs with one loop. On a branched polymer any loop is noncontractible. The "transverse" degree of freedom associated with the a-loops has been eliminated when we consider branched polymers which are made up of elementary links rather than narrow tubes so there is only one loop to study.

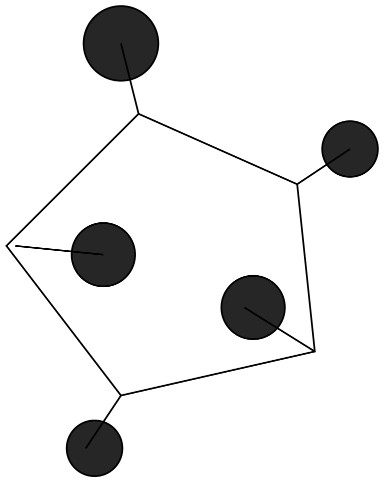

Figure 1: A branched polymer with a single loop of length 5. The blobs stand for the rooted polymer partition function and each of these polymers sits either inside or outside the loop.

We let $Z(\beta)$ denote the partition function for rooted polymers with no loops. Here β is the coupling constant associated with the Boltzmann weight for polymers and we take the action to be the total number of links in the polymer. For simplicity let us assume that the polymers have vertices either of order 1 or 3. This assumption does not affect the values of any generic critical exponents. Let $Z_l(\beta)$ denote the partition function for polymers with a single noncontractible loop of length l. Then

$$Z_l(\beta) = e^{-\beta l} Z(\beta)^l 2^l \tag{14}$$

since any polymer contributing to Z_l can be constructed by gluing rooted polymers to the vertices of a closed loop of length l, see Fig. 1. The factor of 2^l is due to the fact that each rooted polymer can sit inside or outside the loop. It follows that the average length of the loop is given by

$$\bar{l}(\beta) = \frac{\sum_{l=3}^{\infty} l Z_l}{\sum_{l=3}^{\infty} Z_l}. \tag{15}$$

Mimicking the argument used to calculate the two point function for branched polymers [5] it is straightforward to show that

$$Z_l(\beta) \sim e^{-m(\beta) l} \tag{16}$$

as β approaches the critical value β_0 and $m(\beta) \sim (\beta - \beta_0)^{\frac{1}{2}}$ is the mass of the branched polymer model. It follows that that

$$\bar{l}(\beta) \sim (\beta - \beta_0)^{-\frac{1}{2}}. \tag{17}$$

Note that this average is in fact analogous to the average $\bar{l}'$ discussed for random surfaces since the transverse degree is absent for branched polymers.

Above we have presented evidence that generic surfaces contributing to the scaling limit of pure gravity ($c = 0$) are thick in the sense that the expectation value of the length of the shortest noncontractible loop diverges as the critical point is approached. We calculated an analogous quantity for branched polymers and suggested that the collapse of random surfaces to a branched polymer phase is manifested by a symmetry breaking between the length distribution of loops in different homotopy classes.

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