Infinite-bump solutions of the non-linear Schrödinger equation: the non-periodic case

Robert Magnus, Olivier Moschetta

September 2010
RH–19–2010

Infinite-bump solutions of the non-linear Schrödinger equation: the non-periodic case

Robert Magnus*and Olivier Moschetta†

* The University Science Institute, University of Iceland, Dunhaga 3, 107–Reykjavik, Iceland robmag@hi.is
† The University Science Institute, University of Iceland, Dunhaga 3, 107–Reykjavik, Iceland. (Present address) Reykjavik Junior College, Lækjagata 7, 101–Reykjavik, Iceland oms3@hi.is

Abstract

The equation $-\varepsilon^2\Delta u + F(V(x), u) = 0$ is considered in $\mathbb{R}^n$. It is assumed that V possesses a set of critical points B for which the values of V and D^2V satisfy certain compactness and uniformity properties. Under appropriate conditions on F the problem is then shown to possess for each $b \in B$ and small $\varepsilon > 0$ a solution that concentrates at b and has detailed uniformity and decay properties. This implies by results of a previous paper that there exist solutions that concentrate at arbitrary subsets of B as $\varepsilon \to 0$. This includes cases when B is infinite and V non-periodic, instances of which are briefly explored.

1 Introduction

This is the second of two papers on infinite-bump solutions of non-linear Schrödinger-like equations. In a previous paper [4] the authors studied the problem

$$-\varepsilon^2\Delta u + F(V(x), u) = 0, \quad x \in \mathbb{R}^n$$

on the assumption that the function $V(x)$ (the potential function so-called) had a collection B, which could be infinite, of non-degenerate critical points. For each critical point $b \in B$ a solution was assumed to exist for small ε, a so-called single-bump solution, that concentrated at the point b as $\varepsilon \to 0$. This solution was assumed to possess an explicit asymptotic structure as $\varepsilon \to 0$ together with regularity and decay properties that were uniform with respect to ε and $b \in B$. With these conditions it was shown that for arbitrary $B_0 \subset B$ there exists a solution U, in general in the Hölder space $C^{2,\lambda}(\mathbb{R}^n)$, that concentrates at the set B_0 as $\varepsilon \to 0$.

The construction of examples exhibiting infinitely many critical points, with corresponding single bump solutions that satisfy the uniform regularity and decay properties needed to apply these results, is a major problem, and is the main object of this paper. In the previous paper [4], not having presented this construction, we gave a relatively simple illustration of the main theorem in which the potential function was periodic; in fact we restricted ourselves to the case

$$-\varepsilon^2 u'' + V(x)u - u^3 = 0, \quad x \in \mathbb{R}$$

where $V(x)$ is a periodic potential. We then took B as a set of translates of one critical point and used results from [2] for the single bump solution. The uniformity conditions needed to obtain multibump solutions were then trivial consequences of periodicity and the regularity and decay properties for a single solution easy to establish in the context of an ordinary differential equation.

In this paper we derive the uniformity conditions for an infinite collection of single bump solutions using rather simple assumptions that go far beyond the periodic case and allow a much wider range of examples. In effect we revisit the method used in [2] to construct single bump solutions and coerce it into producing uniformity results. We establish that if V has bounded derivatives up to order 3, the crucial property of the set B is that the Hesse matrices $D^2V(b)$ for $b \in B$ are uniformly invertible. The uniformity conditions are then satisfied and we obtain infinite-bump solutions concentrating at arbitrary subsets of B_0 using the glueing-together procedure of [4], although this does seem to require V to have additional bounded derivatives up to order $2 + n/2$. The only real restriction on the construction of examples is then the existence of the ground state as explained in the next section.

In the last section we give some instances based on elementary calculations of non-periodic functions $V(x)$ where the set B is infinite and satisfies our conditions.

As usual we make the change of variable $x = \varepsilon y$ so that we shall, from now on, only consider the problem in the form

$$-\Delta u + F(V(\varepsilon x), u) = 0, \quad x \in \mathbb{R}^n.$$

2 Assumptions

In this section we collect the main assumptions that underlie the existence of the single-bump solutions. These are similar to those of [2] and will ensure the existence of single-bump solutions with uniform properties in H^2. We shall then have to work to establish uniform properties in the Hölder spaces. To avoid unnecessary difficulties we shall assume that F is C^∞. This saves painstaking counting of derivatives. Local differentiabilty of solutions is then not a problem and we can concentrate our efforts on proving global bounds for derivatives and Hölder seminorms.

A note on the function spaces and notation. Generally we consider functions over the whole of $\mathbb{R}^n$. Thus the notations H^k, L^p, C^k, $C^{k,\lambda}$ denote the usual function spaces of functions in $\mathbb{R}^n$ with finite Sobolev or Hölder norms as appropriate. Occasionally we shall need spaces of functions on a closed ball K and use a notation such as $C^k(K)$. Sequences of functions will often be indexed by ν, which runs through the positive integers ("n" is reserved for the dimension of $\mathbb{R}^n$). Local convergence is sometimes used; for example we say $f_\nu \to f$ in C^k_{loc} when $f_\nu \to f$ in the space $C^k(B(0,r))$ for every $r > 0$.

Properties of F

We assume that F is a C^∞ function of $(a,u) \in I \times \mathbb{R}$, where $I \subset \mathbb{R}$ is an open interval. We impose growth conditions:

(F1)
$$\begin{aligned}
|F(a,u)|,\ \left|\frac{\partial F}{\partial a}\Big(a,u\Big)\right|,\ \left|\frac{\partial^2 F}{\partial a^2}\Big(a,u\Big)\right| &\leq C(|u| + |u|^{\alpha_1}),\\
\left|\frac{\partial F}{\partial u}\Big(a,u\Big)\right|,\ \left|\frac{\partial^2 F}{\partial u \partial a}\Big(a,u\Big)\right| &\leq C(1 + |u|^{\alpha_2}),\\
\left|\frac{\partial^2 F}{\partial u^2}\Big(a,u\Big)\right| &\leq C(1 + |u|^{\alpha_3}),
\end{aligned}$$

where the constant C can be chosen uniformly for a in a bounded interval and the exponents are non-negative and satisfy $\alpha_k \geq 2 - k$.

These are called *standard growth conditions* if, in addition, $1 \leq n \leq 7$, with no upper limit placed on $\alpha_1, \alpha_2, \alpha_3$ if $n \leq 4$, whereas for $n = 5, 6, 7$ we assume

$$\alpha_1 < \frac{n}{n-4}, \quad \alpha_2 \leq \frac{4}{n-4}, \quad \alpha_3 < \frac{8-n}{n-4}.$$

Under these conditions the Nemitskii operators defined by the partial derivatives F, $\frac{\partial F}{\partial a}$, $\frac{\partial^2 F}{\partial a^2}$, $\frac{\partial F}{\partial u}$, $\frac{\partial^2 F}{\partial u \partial a}$ and $\frac{\partial^2 F}{\partial u^2}$, enjoy some important boundedness and convergence properties from $L^\infty \times H^2$ to L^2. These are given in [2] pages 588–592.

Properties of V

(V1) V is C^∞ with range in the interval I.

(V2) $D^\alpha V$ is bounded for $|\alpha| \leq 3$.

Positivity property

(P1) There exists $\delta > 0$ such that $\dfrac{\partial F}{\partial u}\big(a, 0\big) > \delta$ for all $a \in I$.

The ground state

For each $a \in I$ we assume the existence of a privileged non-trivial solution ϕ_a in H^2, the ground state, to the equation $-\Delta u + F(a, u) = 0$. This has the following properties.

(Φ1) $\phi_a(x) = \Phi_a(|x|)$ is spherically symmetric.

(Φ2) The map $I \ni a \mapsto \phi_a$ is continuous from I to H^2.

(Φ3) The operator $-\triangle + \frac{\partial F}{\partial u}(a, \phi_a(x)) : H^2 \to L^2$ has as kernel the space spanned by the n partial derivatives $D_j\phi_a(x)$, (which are automatically independent by radiality of ϕ_a), and its range is the space orthogonal in L^2 to its kernel. This property is referred to as quasi-non-degeneracy.

(Φ4) $\displaystyle\int \frac{\partial F}{\partial a}\Big(a, \Phi_a(|x|)\Big)\Phi_a'(|x|)|x|\, dx \neq 0.$

There is an alternative to (Φ4) which is rather appealing; namely that the derivative $D_a\phi$ lies in H^2 and that $(d/da)\int |\nabla\phi_a(x)|^2\, dx \neq 0$. That this implies (Φ4) was shown in [3] and results from the equality

$$\int \frac{\partial F}{\partial a}\Big(a, \Phi_a(|x|)\Big)\Phi_a'(|x|)|x|\, dx = -\frac{d}{da}\left(\int |\nabla\phi_a|^2\, dx\right) \tag{1}$$

3 Properties of the ground state

In this section we deduce some regularity and decay properties of ϕ_a, some of which will be needed for the construction of single bump solutions with uniform properties, while others are interesting in their own right.

Throughout this section, we shall assume without further explicit mention that $F(a, u)$, V and ϕ_a satisfy all the assumptions set out in section 2.

We will repeatedly need a global version of the elliptic regularity theorem.

Theorem 1. *Let $0 < \lambda < 1$, let g be a bounded measurable function on $\mathbb{R}^n$, and let u be a bounded measurable function that satisfies*

$$\Delta u = g$$

in the sense of distributions. Then u belongs to $C^{1,\lambda}(\mathbb{R}^n)$ and satisfies

$$||u||_{C^{1,\lambda}} \leq C(||g||_{L^\infty} + ||u||_{L^\infty})$$

where the constant C depends only on n and λ. If, in addition, g belongs to $C^\lambda(\mathbb{R}^n)$ then $u \in C^{2,\lambda}(\mathbb{R}^n)$ and we have an estimate

$$||u||_{C^{2,\lambda}} \leq C(||g||_{C^\lambda} + ||u||_{L^\infty}).$$

Sometimes we shall claim a result "by interpolation". This means use of the following inequality.

Theorem 2. *Let Ω be either the whole of $\mathbb{R}^n$ or the complement of a closed ball. Let $u \in C^2(\Omega)$. Then*

$$\|Du\|_{L^\infty(\Omega)} \leq 2\|u\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|D^2u\|^{\frac{1}{2}}_{L^\infty(\Omega)}.$$

The main regularity results are as follows.

Theorem 3. *Let $W(x)$ be a C^∞ function with $W \in C^k(\mathbb{R}^n)$ and such that its range lies in a compact subset of the interval I. Let v be a solution in H^2 of*

$$-\Delta v + F(W(x), v) = 0$$

Then $v \in C^{k+1,\lambda}(\mathbb{R}^n)$, for all $0 < \lambda < 1$ and decays at infinity. Moreover there exists a polynomial $Z(\alpha, \beta, \gamma)$, with coefficients depending only on n, k and λ, such that

$$||v||_{C^{k+1,\lambda}} \leq Z\Big(||F||_{C^k(K)}, ||W||_{C^k}, ||v||_{H^2}\Big)||v||_{H^2},$$

where $K = \{(a, u) : |a| \leq ||W||_{C^0}, |u| \leq C_0||v||_{H^2}\}$ and the constant C_0 depends only on n and the constants in the growth condition on $F(a, u)$.

Proof. First assume that $n > 4$ (so that $n = 5$, 6 or 7). By the growth conditions, the Calderon-Zygmund estimate and the Sobolev embedding we find that if $v \in W^{2,r}(\mathbb{R}^n)$ for some r in the range $2 \leq r < n/2$ then $v \in W^{2,nr/\alpha_1(n-2r)}(\mathbb{R}^n)$. Since $v \in W^{2,2}(\mathbb{R}^n)$ we find after a finite number of steps that $v \in W^{2,s}$ for some $s \geq n/2$. In the cases $1 \leq n \leq 4$ this is already known. If $s > n/2$ the Sobolev embedding gives that v is continuous, tends to 0 at infinity and satisfies $||v||_{L^\infty} \leq C_0||v||_{H^2}$ where C_0 depends only on n and the constants in the growth condition on $F(a, u)$. If $s = n/2$ then certainly $v \in L^q$ for all $q \geq 2$ by the Sobolev embedding, and by the equation we find also $v \in W^{2,q}$. Hence again v is continuous, decaying at infinity and satisfies $||v||_{L^\infty} \leq C_0||v||_{H^2}$.

Now we note that v is a bounded solution, $u = v$ of

$$-\Delta u + u = v - F(W(x), v) \tag{2}$$

and the right-hand side is bounded. Hence $v \in C^{1,\lambda}$ by theorem 1. By (2), if $v \in C^{j,\lambda}$ and $j \leq k - 1$ then the right-hand side is in $C^{j,\lambda}$ (since W has enough bounded derivatives to guarantee this) and we deduce that $v \in C^{j+2,\lambda}$. Starting with $v \in C^{1,\lambda}$ we end up at

$v \in C^{k+1,\lambda}$. By induction $||v||_{C^{k+1,\lambda}}$ is bounded by a polynomial, of the required form, in the quantities $||F||_{C^k(K_0)}$, $||W||_{C^k}$ and $||v||_{L^\infty}$, where $K_0 = \{(a,u) : |a| \leq ||W||_{C^0}, |u| \leq ||v||_{L^\infty}\}$ and the coefficients of the polynomial depend only on n, k and λ. We use the inequality $||v||_{L^\infty} \leq C_0||v||_{H^2}$ to replace K_0 by K and complete the proof. □

In a series of lemmas and theorems we study the local and global regularity of the functions ϕ_a, the map $a \mapsto \phi_a$ with various topologies on the codomain (recall assumption (Φ2) according to which it is continuous to H^2) and the quasi-non-degeneracy of ϕ_a in Hölder spaces.

Theorem 4. *The ground state ϕ_a is C^∞. Its derivatives of all orders are bounded on $\mathbb{R}^n$ and decay exponentially at infinity. Moreover the rate of exponential decay is uniform in the following sense: if A is a compact subset of I and α a multi-index there exist $C > 0$ and $\mu > 0$ such that $|D^\alpha\phi_a(x)| \leq Ce^{-\mu|x|}$ for all $x \in \mathbb{R}^n$ and $a \in A$.*

Proof. Here we take W as the constant function a. We therefore have a uniform bound on $||W||_{C^k} = |a|$ for $a \in A$, where $A \subset I$ is compact. Since, by assumption (Φ2), the map $a \mapsto \phi_a : I \to H^2$ is continuous, we have a uniform bound on $||\phi_a||_{H^2}$ for $a \in A$. Hence by theorem 1 every derivative of ϕ_a is bounded, with a bound independent of $a \in A$, and ϕ_a decays at infinity.

We can show that the decay of $\phi_a(x)$ to 0 as $|x| \to \infty$ is uniform with respect to $a \in A$. Suppose that the decay of ϕ_a is not uniform with respect to a. Then we can find $r > 0$ and sequences $a_\nu \in A$ and $x_\nu \in \mathbb{R}^n$ such that $|x_\nu| \to \infty$ and $|\phi_{a_\nu}(x_\nu)| > r$. Since $D\phi_a$ has a uniform bound independent of a we can find $\delta > 0$ such that $|\phi_{a_\nu}(y)| > r/2$ for $|y - x_\nu| < \delta$. Then $\int_{|y-x_\nu|<\delta} |\phi_{a_\nu}(y)|^2\, dy > (r^2/4)\mathrm{vol}\,(B(0,\delta))$. However the set of functions ϕ_a, $a \in A$, is compact in L^2 by assumption (Φ2). So the integral $\int_{|x|>R} |\phi_a|^2\, dx$ converges to 0 as $R \to \infty$, uniformly for $a \in A$. This gives a contradiction since it implies that $\int_{|y-x_\nu|<\delta} |\phi_{a_\nu}(y)|^2\, dy$ tends to 0 as $\nu \to \infty$.

Let $G(a,u) = F(a,u)/u$. Then $\liminf_{|x|\to\infty} G(a, \phi_a(x)) \geq \delta > 0$ (where δ was defined in assumption (P1)), and what is more there exists $R > 0$ such that for $|x| > R$ and all $a \in A$ we have $G(a, \phi_a(x)) \geq \delta/2 > 0$. It follows by [5] that there exist $C > 0$ and $\mu > 0$ such that $|\phi_a(x)| \leq Ce^{-\mu|x|}$ for all $x \in \mathbb{R}^n$ and $a \in A$. By interpolation a similar estimate holds for each derivative $D^\alpha\phi_a$ (but the constant C and exponent μ may depend on α). □

Theorem 5. *Let $0 < \lambda < 1$. The kernel of the operator*

$$A_a := -\Delta + \frac{\partial F}{\partial u}\Big(a, \phi_a\Big) : C^{2,\lambda} \to C^{0,\lambda}$$

is spanned by the partial derivatives $D_j\phi_a$, $j = 1, \ldots, n$ and its range is the orthogonal L^2-complement in $C^{0,\lambda}$ of its kernel. The same result applies to the operator acting from $C^{k+2,\lambda}$ to $C^{k,\lambda}$ for $k = 1, 2, \ldots$.

Proof. The proof is in three steps.

(A) *Identification of the kernel of A_a.*

Let $v \in C^{2,\lambda}$ be in the kernel of A_a. Then

$$-\Delta v + \frac{\partial F}{\partial u}(a,0)v + \left(\frac{\partial F}{\partial u}(a,\phi_a) - \frac{\partial F}{\partial u}(a,0)\right) v = 0.$$

Since ϕ_a decays exponentially, and since

$$\frac{\partial F}{\partial u}(a,\phi_a) - \frac{\partial F}{\partial u}(a,0) = \phi_a \int_0^1 \frac{\partial^2 F}{\partial u^2}(a, t\phi_a)\, dt$$

and v is bounded we have that

$$\left(\frac{\partial F}{\partial u}(a,\phi_a) - \frac{\partial F}{\partial u}(a,0)\right) v$$

belongs to L^2. Thus we have both that v is bounded and $(-\Delta + m)v \in L^2$ where $m = \frac{\partial F}{\partial u}(a,0) > 0$. We deduce that $v \in H^2$. But now ($\phi 3$) implies that v is in the space spanned by the partial derivatives $D_j\phi_a$.

(B) A_a *is a Fredholm operator of index* 0. A_a is a compact perturbation of the operator $-\Delta + \frac{\partial F}{\partial u}\big(a,0\big) : C^{2,\lambda} \to C^{0,\lambda}$ and the latter is invertible and surjective in view of the condition $\frac{\partial F}{\partial u}\big(a,0\big) > \delta > 0$. (This is a place where Hölder spaces must be used; the last claim is untrue without the Hölder exponent.)

(C) *Identification of the range of A_a.* It is straightforward to check that $A_a u$ is orthogonal to $D_j\phi_a$. By Fredholmness the range is exactly its L^2-orthogonal complement in $C^{0,\lambda}$.

□

Theorem 6. *The map $a \mapsto \phi_a$ is continuous from I to C^k and from I to H^k for all integers $k \geq 0$.*

Proof. We first consider the map from I to C^0. We first remark that if $A \subset I$ is compact we have a uniform bound on $D\phi_a(x)$ independent of $a \in A$; hence the family ϕ_a, $a \in A$, is uniformly equicontinuous. Now let $a_\nu \to a$. We must show that $\phi_{a_\nu} \to \phi_a$ uniformly in $\mathbb{R}^n$. If this is not the case then, going to a subsequence if necessary, we can find a sequence $x_\nu \in \mathbb{R}^n$ such that $|\phi_{a_\nu}(x_\nu) - \phi_a(x_\nu)| > \alpha$ where $\alpha > 0$. Moreover since $\phi_{a_\nu} \to \phi_a$ in L^2 we may suppose that $\phi_{a_\nu} \to \phi_a$ a.e. Then by equicontinuity we must have $\phi_{a_\nu} \to \phi_a$ pointwise (if not then the set of points at which ϕ_{a_ν} does not converge to ϕ_a is open). Suppose first that x_ν is bounded. Then we may assume that it converges to x say. Using equicontinuity (the above remark with $A = \{a_\nu\} \cup \{a\}$), we have that $\phi_{a_\nu}(x_\nu) - \phi_{a_\nu}(x) \to 0$, $\phi_a(x_\nu) - \phi_a(x) \to 0$ and $\phi_{a_\nu}(x) - \phi_a(x) \to 0$. This contradicts the inequality $|\phi_{a_\nu}(x_\nu) - \phi_a(x_\nu)| > \alpha$. If, on the other hand, x_ν is unbounded we may assume that $|x_\nu| \to \infty$. But this contradicts the uniform decay of ϕ_a at infinity.

Now it follows by interpolation that $a \mapsto \phi_a$ is continuous from I to $C^k(\mathbb{R}^n)$.

Finally we study the map $a \mapsto \phi_a$ from I to H^k. Let α be a multi-index. We must show that

$$\lim_{\nu\to\infty} \int |D^\alpha \phi_a(x) - D^\alpha \phi_{a_\nu}(x)|^2\, dx = 0$$

if $a_\nu \to a$. This follows from uniform exponential decay of the derivatives of ϕ_a, pointwise convergence to 0 of the integrand and the dominated convergence theorem. □

It is convenient here to present some preliminary lemmas needed for the construction of single bump solutions with uniform properties. Let T be an $n \times n$ matrix. Then the function $\frac{\partial F}{\partial a}(a, \phi_a) Tx \cdot x$ is orthogonal to the partial derivatives $D_j \phi_a$ by radiality and belongs to L^2 by exponential decay of ϕ_a. Hence the equation

$$-\Delta v + \frac{\partial F}{\partial u}(a, \phi_a) v = \frac{\partial F}{\partial a}(a, \phi_a) Tx \cdot x$$

has a unique solution in H^2. In particular for each $b \in \mathbb{R}^n$ we may define η^b as the unique solution in H^2 orthogonal to the partial derivatives $D_j\phi_a$ of the equation

$$-\Delta v + \frac{\partial F}{\partial u}(V(b), \phi_{V(b)})v = -\frac{1}{2}\frac{\partial F}{\partial a}(V(b), \phi_{V(b)}) D^2V(b)x \cdot x. \tag{3}$$

The next two propositions establish some uniform properties of the family η^b.

Theorem 7. *Let χ^a_{ij} be the unique solution in H^2, orthogonal in the L^2-sense to the partial derivatives $D_j\phi_a$, of the equation*

$$-\Delta v + \frac{\partial F}{\partial u}(a, \phi_a) v = -\frac{1}{2}\frac{\partial F}{\partial a}(a, \phi_a) x_i x_j$$

Then the map $a \mapsto \chi^a_{ij}$ is continuous from I to H^2.

Proof. Firstly let $A_a : H^2 \to L^2$ be the operator $-\Delta + \frac{\partial F}{\partial u}(a, \phi_a)$. By theorem 6, in particular the continuity of $a \mapsto \phi_a$ in the C^0 topology, the map $a \mapsto A_a$ is continuous in the operator-norm topology.

Let $P^a_j \in (L^2)^*$ be the linear functional $P^a_j v = \int v D_j \phi_a\, dx$. By the continuity of $a \mapsto D_j\phi_a$ from I to L^2 as proved above, the map $a \mapsto P^a_j$ is continuous in the norm topology.

The map $a \mapsto \frac{1}{2}\frac{\partial F}{\partial a}(a, \phi_a) x_i x_j$ is continuous in the L^2-topology, by uniform exponential decay of ϕ_a and the growth conditions (F1).

Define the linear map N_a from $H^2 \oplus \mathbb{R}^n$ to $L^2 \oplus \mathbb{R}^n$

$$N_a(v, s) = (A_a v + \nabla\phi_a \cdot s,\ (P^a_j v)^n_{j=1})$$

By the properties of A_a the map N_a is invertible and depends continuously on a in the operator-norm topology. Hence the inverse N_a^{-1} also depends continuously on a. But χ^a_{ij} is determined by the operator equation

$$N_a(\chi^a_{ij}, 0) = \Big(-\frac{1}{2}\frac{\partial F}{\partial a}\big(a, \phi_a\big)x_i x_j,\ 0\Big)$$

or equivalently χ^a_{ij} is the first coordinate of

$$N_a^{-1}\Big(-\frac{1}{2}\frac{\partial F}{\partial a}\big(a, \phi_a\big)x_i x_j,\ 0\Big).$$

Hence the map $a \mapsto \chi^a_{ij}$ is continuous in the H^2-topology. □

Lemma 8. *Let $B \subset \mathbb{R}^n$ be such that the set $V(B)$ has compact closure in I. Then the set of functions $\{\eta^b : b \in B\}$, is relatively compact in H^2 and is a bounded family in C^k for every k.*

Proof. The first part follows by writing

$$\eta^b = \sum_{ij} \chi^{V(b)}_{ij} D_{ij} V(b)$$

whence we deduce by assumption $(V2)$, condition (1) of theorem 11 and theorem 7 that the family η^b, $b \in B$, is relatively compact in H^2.

The second part follows by treating the family χ^a_{ij} as a family in $C^{k,\lambda}$. We view N_a as a map from $C^{k+2,\lambda} \oplus \mathbb{R}^n$ to $C^{k,\lambda} \oplus \mathbb{R}^n$ and χ^a_{ij} as the inverse image of $(-\frac{1}{2}\frac{\partial F}{\partial a}\big(a, \phi_a\big)x_i x_j, 0)$. □

4 Existence principles

We formulate here the abstract existence principles that we shall use. The following is essentially taken from [2].

Theorem 9. *Let E and F be real Banach spaces, and let $f : \mathbb{R}_+ \times E \to F$ and let $x_0 \in E$. Assume that*

(1) $f(\varepsilon, \cdot)$ is C^2 for each $\varepsilon \geq 0$;

(2) $\lim_{\varepsilon \to 0^+} f(\varepsilon, x_0) = 0$;

(3) for all sufficiently small $\varepsilon > 0$ the operator $D_x f(\varepsilon, x_0)$ is invertible and $||D_x f(\varepsilon, x_0)^{-1}||$ is uniformly bounded as $\varepsilon \to 0^+$;

(4) $D^2_x f(\varepsilon, x)$ is bounded in the operator norm given that ε is bounded above and x is

restricted to a bounded subset of E.

Then there exist $\varepsilon_0 > 0$ and a neighbourhood U of x_0 in E such that for each ε in the range $0 < \varepsilon < \varepsilon_0$ there exists a unique solution $x = x_\varepsilon$ of $f(\varepsilon, x) = 0$ in U. Moreover $\lim_{\varepsilon\to 0} x_\varepsilon = x_0$.

If, furthermore, $f(\varepsilon, x)$ is a continuous function of ε for $\varepsilon > 0$ and for all x in a neighbourhood of x_0, and the map $\varepsilon \mapsto D_x f(\varepsilon, x_0)$ is continuous in the strong operator topology*, then the solution x_ε depends continuously on ε.*

The following essentially trivial result will be repeatedly invoked at key places (Wang's lemma [2]).

Lemma 10. *Let f_ν be a family of measurable functions such that*

$$0 < \delta < f_\nu(x) < K$$

for all ν and constants δ and K. Let μ_ν be a sequence of non-negative numbers and let v_ν be a sequence in H^2 such that

$$-\Delta v_\nu + (f_\nu(x) + \mu_\nu)v_\nu \to 0$$

in L^2. Then $v_\nu \to 0$ in H^2.

5 Single bump solutions

In this section we construct a family of single bump solutions, with some uniformity properties that can be expressed in H^2. Recall the family of functions η^b defined in section 3 as the unique solution to (3) orthogonal to the partial derivatives $D_j\phi_{V(b)}$.

Theorem 11. *Assume all the conditions listed in section 2. Let B be a collection of non-degenerate critical points of V. Assume that:*

(1) *the set of values $V(b)$, $b \in B$, has compact closure in I;*

(2) *the collection of symmetric matrices $D^2V(b)$, $b \in B$, has the property* $|\det D^2V(b)| > h$ *where $h > 0$ and is independent of $b \in B$.*

Then there exists ε_0 and a family of numbers $r_\varepsilon > 0$ such that $\lim_{\varepsilon\to 0+} r_\varepsilon = 0$*, and for* $0 < \varepsilon < \varepsilon_0$ *and each $b \in B$ the problem*

$$-\Delta u + F(V(\varepsilon x), u) = 0$$

has a unique solution $u_\varepsilon^b \in H^2$ having the form

$$u_\varepsilon^b(x) = \phi_{V(b)}\Big(x - \frac{b}{\varepsilon} + s_\varepsilon^b\Big) + \varepsilon^2 w_\varepsilon^b\Big(x - \frac{b}{\varepsilon} + s_\varepsilon^b\Big)$$

where $s^b_\varepsilon \in \mathbb{R}^n$, the function w^b_ε is orthogonal to the partial derivatives $D_j\phi_{V(b)}$, and $|s^b_\varepsilon| + ||w^b_\varepsilon - \eta^b||_{H^2} < r_\varepsilon$.
Finally the family of maps $\varepsilon \mapsto (s^b_\varepsilon, w^b_\varepsilon)$, $b \in B$, from $]0, \epsilon_0[$ to $\mathbb{R}^n \times H^2$ is equicontinuous.

Proof. For each $a \in I$ let

$$W^a = \Big\{ w \in H^2 : \int w D_j \phi_a \, dx = 0, \ j = 1, \dots, n \Big\}.$$

We seek a solution of the form

$$u(x) = \phi_a\Big(x - \frac{b}{\varepsilon} + s\Big) + \varepsilon^2 w\Big(x - \frac{b}{\varepsilon} + s\Big),$$

where $b \in B$, $a = V(b)$, $s \in \mathbb{R}^n$ and $w \in W^a$. It is sometimes convenient to write this as

$$u(x) = \phi_a(x - \xi) + \varepsilon^2 w(x - \xi)$$

where $\xi = \frac{b}{\varepsilon} - s$. Substituting into the problem, using $-\triangle\phi_a(x) + F(a, \phi_a(x)) = 0$ and replacing x by $x + \xi$ we obtain

$$-\varepsilon^2 \triangle w - F(a, \phi_a) + F\big(V(\varepsilon(x + \xi)), \phi_a + \varepsilon^2 w\big) = 0,$$

Dividing by ε^2 we form the *rescaled equation*

$$-\triangle w + \varepsilon^{-2}\Big[F\big(V(\varepsilon(x + \xi)), \phi_a + \varepsilon^2 w\big) - F\big(a, \phi_a\big)\Big] = 0. \tag{4}$$

This problem can be considered an operator equation

$$\Gamma^b_\varepsilon(s, w) = 0 \tag{5}$$

where $\Gamma^b_\varepsilon : \mathbb{R}^n \times W^a \to L^2$. As usual we use the convention $a = V(b)$.

Essentially we shall solve (5) for small ε uniformly for $b \in B$. To see what this entails expand (4) into

$$- \triangle w + \varepsilon^{-2}\Big[F\big(V(\varepsilon(x + \xi)), \phi_a + \varepsilon^2 w\big) - F\big(V(\varepsilon(x + \xi)), \phi_a\big)\Big] \\ + \varepsilon^{-2}\Big[F\big(V(\varepsilon(x + \xi)), \phi_a\big) - F\big(a, \phi_a\big)\Big] = 0. \tag{6}$$

and for each $b \in B$ consider the (non-rigorous) limit as $\varepsilon \to 0$

$$-\triangle w + \frac{\partial F}{\partial u}\big(a, \phi_a\big) w + \frac{1}{2}\frac{\partial F}{\partial a}\big(a, \phi_a\big)\big(D^2 V(b)(x - s) \cdot (x - s)\big) = 0 \tag{7}$$

which has the unique solution $s = 0$, $w = \eta^b$.

We now claim:

(1) $\Gamma^b_\varepsilon(0,\eta^b) \to 0$ in L^2 as $\varepsilon \to 0$, uniformly with respect to $b \in B$.

(2) $D\Gamma^b_\varepsilon(0,\eta^b)$ is an invertible, surjective, linear operator from $\mathbb{R}^n \times W^a$ to L^2 and its inverse is uniformly bounded in the operator norm with respect to a range of values of ε of the form $0 < \varepsilon < \varepsilon_1$ and all $b \in B$.

(3) $D^2\Gamma^b_\varepsilon(s,w)$ and $D^3\Gamma^b_\varepsilon(s,w)$ are uniformly bounded in norm given a uniform bound for the norm of (s,w), an upper bound for ε and no restriction on $b \in B$.

All our conclusions except the last one follow from these claims by means of theorem 9 together with a simple device. Let E denote the subspace of the cartesian product $\Pi_{b\in B}(\mathbb{R}^n \times W^{V(b)})$ consisting of those families $(s^b,w^b)_{b\in B}$ that are uniformly bounded with respect to $b \in B$. We impose the norm $||(s^b,w^b)_{b\in B}|| = \sup_{b\in B}\left(|s^b| + ||w^b||\right)$. In effect E is just an l^∞ space of sequences indexed by B where the bth coordinate of each sequence lies in $\mathbb{R}^n \times W^{V(b)}$. We also introduce the space F of bounded families of elements of L^2 indexed by B. Then we define an operator $\Gamma^B_\varepsilon : E \to F$ by

$$\Gamma^B_\varepsilon((s^b,w^b)_{b\in B}) = (\Gamma^b_\varepsilon(s^b,w^b))_{b\in B}.$$

We will apply theorem 9 to solve the operator equation

$$\Gamma^B_\varepsilon((s^b,w^b)_{b\in B}) = 0 \tag{8}$$

By claim (3) Γ^B_ε is C^2 and the second derivative $D^2\Gamma^B_\varepsilon((s^b,w^b)_{b\in B})$ is uniformly bounded given that $(s^b,w^b)_{b\in B}$ is restricted to a bounded subset of E and an upper bound placed on ε. By lemma 8 the family $(0,\eta^b)_{b\in B}$ belongs to E. By claim (1) $\lim_{\varepsilon\to 0}\Gamma^B_\varepsilon((0,\eta^b)_{b\in B}) = 0$ and by claim (2) the operator $D\Gamma^B_\varepsilon((0,\eta^b)_{b\in B})$ is invertible for all sufficiently small ε and its inverse is bounded in norm as $\varepsilon \to 0$. All the conclusions of the theorem except the last now follow from theorem 9.

Now to proving the claims (1), (2) and (3).

Claim (1) follows by expressing the difference between (6) and (7) as integrals, using the boundedness properties of the Nemitski-operators induced by the derivatives of F together with the facts that the family ϕ_a has uniform exponential decay for $a = V(b) \in V(B)$, that B consists of critical points of V and η^b is uniformly bounded in H^2 for $b \in B$ (lemma 8).

Claim (3) follows, much as in the case of claim (1), by expressing the left-hand side of (4) by integrals to eliminate negative powers of ε.

To prove claim (2) we note first that $D\Gamma^b_\varepsilon(0,\eta^b)$ is a Fredholm operator of index 0 as it is a compact perturbation of the invertible operator $-\Delta + \frac{\partial F}{\partial u}(a,0)$. It is sufficient therefore to prove the following.

(4) Let b_ν be a sequence in B, let $a_\nu = V(b_\nu)$, let $\varepsilon_\nu \to 0$ and let $(\sigma_\nu, v_\nu) \in \mathbb{R}^n \times W^{a_\nu}$ satisfy $|\sigma_\nu| + ||v_\nu||_{H^2} \le 1$ and $||D\Gamma^{b_\nu}_{\varepsilon_\nu}(0,\eta^{b_\nu})(\sigma_\nu,v_\nu)||_{L^2} \to 0$. Then a subsequence of (σ_ν, v_ν)

tends to 0 in $\mathbb{R}^n \times H^2$.

Given the sequences in the first sentence of claim (4) we may assume, going to a subsequence, that $a_\nu \to a_0 \in I$ (by condition (1) of the theorem), $\sigma_\nu \to \sigma_0 \in \mathbb{R}^n$ and $v_\nu \to v_0$ weakly in H^2. Moreover we may assume, by condition (2) of the theorem, that $D^2V(b_\nu)$ converges to a symmetric matrix A_0 which will be invertible. By assumption (Φ2) $\phi_{a_\nu} \to \phi_{a_0}$ in H^2.

The relation $||D\Gamma_{\varepsilon_\nu}^{b_\nu}(0,\eta^{b_\nu})(\sigma_\nu, v_\nu)|| \to 0$ expands into

$$\begin{aligned}
&- \triangle v_\nu + \frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu x + b_\nu), \phi_{a_\nu} + \varepsilon_\nu^2 \eta^{b_\nu}\Big) v_\nu \\
&- \varepsilon_\nu^{-1} \left(\frac{\partial F}{\partial a}\Big(V(\varepsilon_\nu x + b_\nu), \phi_{a_\nu} + \varepsilon_\nu^2 \eta^{b_\nu}\Big) - \frac{\partial F}{\partial a}\Big(V(\varepsilon_\nu x + b_\nu), \phi_{a_\nu}\Big) \right) \nabla V(\varepsilon_\nu x + b_\nu) \cdot \sigma_\nu \\
&\qquad\qquad - \varepsilon_\nu^{-1} \frac{\partial F}{\partial a}\Big(V(\varepsilon_\nu x + b_\nu), \phi_{a_\nu}\Big) \nabla V(\varepsilon_\nu x + b_\nu) \cdot \sigma_\nu \to 0. \quad (9)
\end{aligned}$$

Proceeding to the limit we find that the following equation is satisfied in the distribution sense:

$$-\Delta v_0 + \frac{\partial F}{\partial u}\Big(a_0, \phi_{a_0}\Big) v_0 - \frac{\partial F}{\partial a}\Big(a_0, \phi_{a_0}\Big) A_0 x \cdot \sigma_0 = 0 \qquad (10)$$

As in the previous paper [2] the invertibility of A_0 and the non-vanishing of the integral (assumption (Φ5)) imply that $\sigma_0 = 0$. We also know that

$$\int v_\nu D_j \phi_{a_\nu}\, dx = 0, \quad j = 1, \dots, n$$

for each ν; so going to the limit gives

$$\int v_0 D_j \phi_{a_0}\, dx = 0, \quad j = 1, \dots, n$$

so that we deduce from (10) that $v_0 = 0$. We therefore deduce from (9) that

$$-\triangle v_\nu + \frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu x + b_\nu), \phi_{a_\nu} + \varepsilon_\nu^2 \eta^{b_\nu}\Big) v_\nu \to 0 \qquad (11)$$

As in the previous paper ([2], lemma 3.5(i)) this implies that

$$-\triangle v_\nu + \frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu x + b_\nu), 0\Big) v_\nu \to 0 \qquad (12)$$

which combined with lemma 10 gives $v_\nu \to 0$ in H^2. This proves claim (4).

Finally we consider the equicontinuity of the family of maps $\varepsilon \mapsto (s_\varepsilon^b, w_\varepsilon^b)$, $b \in B$. We wish to apply the last claim of theorem 9. The hard part is continuity with respect to the strong operator topology; we have to show that the map

$$\varepsilon \mapsto D\Gamma_\varepsilon^B\big((0,\eta^b)_{b\in B}\big)(\sigma^b, v^b)_{b\in B}$$

is continuous from $]0,\varepsilon_0[$ to the space F, where $(\sigma^b, v^b)_{b\in B}$ is a fixed element of E. This amounts to showing that the family of maps

$$\varepsilon \mapsto D\Gamma^b_\varepsilon(0,\eta^b)(\sigma^b, v^b), \quad b \in B$$

is equicontinuous. Unfortunately this cannot be true. Let us expand this mapping into

$$\varepsilon \mapsto -\triangle v^b + \frac{\partial F}{\partial u}\Big(V(\varepsilon x + b), \phi_{V(b)} + \varepsilon^2\eta^b\Big)v^b \\ - \varepsilon^{-1}\frac{\partial F}{\partial a}\Big(V(\varepsilon x + b), \phi_{V(b)} + \varepsilon^2\eta^b\Big)\nabla V(\varepsilon x + b)\cdot\sigma^b \quad (13)$$

The problem is that the second term cannot be equicontinuous as a function of ε for arbitrary bounded families v^b since the limit

$$\lim_{\varepsilon\to\varepsilon_1}\frac{\partial F}{\partial u}\Big(V(\varepsilon x + b), \phi_{V(b)} + \varepsilon^2\eta^b\Big)$$

is not uniform with respect to x. In fact it is because of this problem that we could not apply the normal implicit function theorem to obtain solutions of (8).

The solution is to modify the device used in the preceding proof. Recall that the family $(s^b_\varepsilon, w^b_\varepsilon)_{b\in B}$ is the unique solution of the operator equation $\Gamma^B_\varepsilon((s^b, w^b)_{b\in B}) = 0$ near to the family $(0,\eta^b)_{b\in B}$. Here Γ^B_ε acts in the space E of *uniformly bounded families* of the form $(s^b, w^b)_{b\in B}$ with $w^b \in W^{V(b)}$ and its codomain is the space F of uniformly bounded families $(f^b)_{b\in B}$ in L^2. The idea is to replace the defining property of *uniform boundedness* with the more restrictive property that the families are *relatively compact*. This replaces the spaces E and F by the closed subspaces E_0 and F_0 of relatively compact families. Since we already know by lemma 8 that the family η^b is relatively compact in H^2, and hence also that $(0,\eta^b)_{b\in B} \in E_0$, we can conclude that the family $(s^b_\varepsilon, w^b_\varepsilon)_{b\in B} \in E_0$ if we can carry out the existence argument using E_0 and F_0 instead of E and F.

Now it is simple to show that (13) defines an equicontinuous family when the family v^b, $b \in B$, is relatively compact in L^2 instead of being merely bounded and hence we obtain the final claim of the theorem.

The remaining two tasks are to show

(1) Γ^B_ε maps E_0 into F_0;

(2) $D\Gamma^B_\varepsilon((0,\eta^b)_{b\in B})$ is surjective when viewed as a linear map from E_0 to F_0.

Proof of (1). This requires showing that, given a sequence $b_\nu \in B$, if $w_\nu := w^{b_\nu} \in W^{V(b_\nu)}$ is convergent in H^2 and $s_\nu := s^{b_\nu}$ is convergent in $\mathbb{R}^n$, then the sequence

$$f_\nu = -\triangle w_\nu + \varepsilon^{-2}\Big[F\big(V(\varepsilon(x - s_\nu) + b_\nu), \phi_{V(b_\nu)} + \varepsilon^2 w_\nu\big) - F\big(V(b_\nu), \phi_{V(b_\nu)}\big)\Big]$$

has a convergent subsequence in L^2. This is obtained by choosing a subsequence along which $V(b_\nu)$ is convergent and the sequence $V(\varepsilon(\cdot - s_\nu) + b_\nu)$ is convergent in C_{loc}.

Proof of (2). We refer to the formula for the derivative $D\Gamma_\varepsilon^B((0,\eta^b)_{b\in B})$. It suffices to show the following. Let $(f^b)_{b\in B} \in F$. We know that there is a unique family $(\sigma^b, v^b)_{b\in B} \in E$ such that

$$D\Gamma_\varepsilon^B((0,\eta^b)_{b\in B})(\sigma^b, v^b)_{b\in B} = (f^b)_{b\in B}$$

The problem is to show that if $(f^b)_{b\in B} \in F_0$ then $(\sigma^b, v^b)_{b\in B} \in E_0$. This amounts to showing the following. Given a sequence $b_\nu \in B$ and a sequence $f_\nu \in L^2$ that is convergent to f, let (σ_ν, v_ν) be the unique solution in $\mathbb{R}^n \times W^{V(b_\nu)}$ to

$$\begin{aligned} -\triangle v_\nu + \frac{\partial F}{\partial u}\Big(V(\varepsilon x + b_\nu), \phi_{a_\nu} + \varepsilon^2 \eta^{b_\nu}\Big) v_\nu \\ - \varepsilon^{-1} \frac{\partial F}{\partial a}\Big(V(\varepsilon x + b_\nu), \phi_{a_\nu} + \varepsilon^2 \eta^{b_\nu}\Big) \nabla V(\varepsilon x + b_\nu) \cdot \sigma_\nu = f_\nu. \end{aligned} \tag{14}$$

We wish to show that (σ_ν, v_ν) has a convergent subsequence in $\mathbb{R}^n \times H^2$. By going to a subsequence we may assume that $\sigma_\nu \to \sigma$, $a_\nu \to a$, $V(\varepsilon(\cdot) + b_\nu) \to h$ in C^1_{loc}, $\eta^{b_\nu} \to \eta$ in H^2 and v_ν is weakly convergent in H^2 to a function v. Going to the limit we deduce

$$-\triangle v + \frac{\partial F}{\partial u}\Big(h, \phi_a + \varepsilon^2 \eta\Big) v - \varepsilon^{-1} \frac{\partial F}{\partial a}\Big(h, \phi_a + \varepsilon^2 \eta\Big) \nabla h \cdot \sigma = f. \tag{15}$$

By the convergence lemmas of [2] we deduce

$$-\Delta(v_\nu - v) + \frac{\partial F}{\partial u}\Big(V(\varepsilon x + b_\nu), 0\Big)(v_\nu - v) \to 0$$

in L^2. Now lemma 10 implies that $v_\nu \to v$ in H^2.

This ends the proof of theorem 11. □

It might be asked why we did not carry out the proof of theorem 11 directly in Hölder spaces instead of in Sobolev spaces. This would have obviated the need for growth conditions and granted some of the regularity to be proved in the next section. Against this one can counter the following. Firstly growth conditions are needed anyway to ensure the existence of the ground state, which is usually found by solving an extremal problem in H^1. Secondly our proof follows closely the proof of theorem 3.8 of [2]. Thirdly the finiteness of Sobolev norms (read energy) is a desirable conclusion in itself when it is appropriate as is certainly the case for single bump solutions.

6 Regularity and decay estimates

In this section we give a long list of results whose purpose is to prepare for the glueing together of the solutions u_ε^b obtained in theorem 11 into new solutions. Since B may be infinite this process cannot be carried out in H^2 and instead we must use spaces of classically differentiable functions with global bounds and Hölder conditions. This was studied in a previous paper [4] so that it is only necessary here to show that the solutions u_ε^b have the regularity and uniform decay properties listed in [4].

In the previous paper [4] the authors gave an example in which V was periodic. Then essentially ϕ_a and w_ε^b were independent of b (or had finitely many values only as b varies). The uniformity needed was then trivially obtained.

The next objective is the regularity and uniform exponential decay of the single bump solutions u_ε^b. The crucial result needed to get the uniformity is the relative compactness of the family w_ε^b in H^2. Throughout this section we refer to the number ε_0 introduced in theorem 11.

Lemma 12. *Let B satisfy the conditions of theorem 11 and let $0 < \varepsilon_1 < \varepsilon_0$. Then the family w_ε^b, $0 < \varepsilon < \varepsilon_1$, $b \in B$, is relatively compact in H^2.*

Proof. The family $(w_\varepsilon^b)_{b\in B}$ is relatively compact at fixed ε by the proof of theorem 11. Now we allow ε to move and use equicontinuity of the map $\varepsilon \mapsto w_\varepsilon^b$ as proved in theorem 11. □

Theorem 13. *Let B satisfy the conditions of theorem 11 and let $0 < \lambda < 1$. Then the solutions u_ε^b are C^∞ and decay at infinity. If $0 < \varepsilon_1 < \varepsilon_0$ then the family u_ε^b, $0 < \varepsilon < \varepsilon_1$, $b \in B$, is bounded in $C^{4,\lambda}$.*

Proof. Since V and F are C^∞ it follows that u_ε^b is C^∞. Since V has bounded derivatives up to order 3 then, from the equation

$$-\Delta u_\varepsilon^b + F(V(\varepsilon x), u_\varepsilon^b) = 0$$

we see by theorem 3 that $u_\varepsilon^b \in C^{4,\lambda}$. To get a uniform bound on the norm using theorem 3 we must have a uniform bound in H^2. This requires a uniform bound on the H^2 norm of ϕ_a for $a \in V(B)$ and a uniform bound on $||w_\varepsilon^b||_{H^2}$. The former follows from assumption (Φ2); the latter from lemma 12.

□

Theorem 14. *Let B satisfy the conditions of theorem 11 and let $0 < \lambda < 1$. If $0 < \varepsilon_1 < \varepsilon_0$ then the family w_ε^b is uniformly bounded in $C^{2,\lambda}$ for $0 < \varepsilon < \varepsilon_1$ and $b \in B$.*

Proof. Note that we already have by the last theorem and the asymptotic formula for u_ε^b that w_ε^b is in $C^{4,\lambda}$ and that a uniform bound exists on the $C^{4,\lambda}$ norm of $\varepsilon^2 w_\varepsilon^b$. However there is no reason to suppose in advance that a bound exists on any $C^{k,\lambda}$-norm of w_ε^b independent of ε, except that obtained from the Sobolev embedding of H^2 into $C^{k,\lambda}$, which gives a result only for $n = 1$, 2 and 3, and a much poorer one than the conclusion of this theorem.

We approach the question by viewing w_ε^b as a solution to a semilinear equation. We know that

$$\Delta w_\varepsilon^b = \varepsilon^{-2}\Big[F(V(b+\varepsilon(x-s_\varepsilon^b)), \phi_a + \varepsilon^2 w_\varepsilon^b) - F(a, \phi_a)\Big]$$

where, as usual $a = V(b)$. Write the right-hand side as $I_\varepsilon^b(x) + J_\varepsilon^b(x)$ where

$$\begin{aligned} I_\varepsilon^b(x) &= \varepsilon^{-2}\Big[F(V(b+\varepsilon(x-s_\varepsilon^b)),\phi_a+\varepsilon^2 w_\varepsilon^b) - F(V(b+\varepsilon(x-s_\varepsilon^b)),\phi_a)\Big] \\ &= \int_0^1 \frac{\partial F}{\partial u}\Big(V(b+\varepsilon(x-s_\varepsilon^b)),\phi_a + t\varepsilon^2 w_\varepsilon^b\Big) w_\varepsilon^b\, dt \end{aligned}$$

and

$$\begin{aligned} J_\varepsilon^b(x) &= \varepsilon^{-2}\Big[F(V(b+\varepsilon(x-s_\varepsilon^b)),\phi_a) - F(a,\phi_a)\Big] \\ &= \int_0^1\int_0^1 \frac{\partial F}{\partial a}\Big(V(b+t_1\varepsilon(x-s_\varepsilon^b)),\phi_a\Big) H\big(b+t_1t_2\varepsilon(x-s_\varepsilon^b)\big)(x-s_\varepsilon^b)\cdot(x-s_\varepsilon^b)t_1\, dt_1 dt_2 \end{aligned}$$

Now we know that $\phi_a + t\varepsilon^2 w_\varepsilon^b$ is uniformly bounded in C^0 with respect to $b \in B$ and $0 < \varepsilon < \varepsilon_1$. Also V is bounded. Hence

$$|I_\varepsilon^b(x)| \leq C|w_\varepsilon^b(x)|$$

where C is independent of $\varepsilon < \varepsilon_1$ and $b \in B$.

Turning to J_ε^b we know by assumptions (F1), (V2) and theorem 4 that the family $\frac{\partial F}{\partial a}(V(b + t_1\varepsilon(x - s_\varepsilon^b)), \phi_a)$ has uniform exponential decay w.r.t. $b \in B$, $\varepsilon < \varepsilon_0$ and $t_1 \in]0,1[$. The Hessian matrix H of V is bounded, and the family s_ε^b is bounded for $0 < \varepsilon < \varepsilon_1$ and $b \in B$. Hence there exists $C > 0$, independent of $b \in B$ and $\varepsilon < \varepsilon_1$, such that $|J_\varepsilon^b(x)| \leq C$, and also $||J_\varepsilon^b||_{L^2} \leq C$, and hence $||J_\varepsilon^b||_{L^p} \leq C$ for $2 \leq p \leq \infty$.

Now we can perform a bootstrap starting with $w_\varepsilon^b \in W^{2,2}$, since if $w_\varepsilon^b \in W^{2,r}$ for some r in the range $2 \leq r < n/2$ and $||w_\varepsilon^b||_{W^{2,r}} \leq D$ with D independent of ε and b we find

$$||\Delta w_\varepsilon^b||_{L^{\frac{nr}{n-2r}}} \leq ||I_\varepsilon^b||_{L^{\frac{nr}{n-2r}}} + ||J_\varepsilon^b||_{L^{\frac{nr}{n-2r}}} \leq C\big(||w_\varepsilon^b||_{L^{\frac{nr}{n-2r}}} + 1\big)$$

and the Calderon-Zygmund estimate gives that $||w_\varepsilon^b||_{W^{2,\frac{nr}{n-2r}}}$ has a finite bound independent of ε and b. Applying the bootstrap leads in a finite number of steps to

$$||w_\varepsilon^b||_{L^\infty} \leq C$$

for a (new) constant C independent of $\varepsilon < \varepsilon_1$ and b. (Note that for $n = 1$, 2, 3 or 4 the bootstrap is unnecessary and we have this conclusion at once.) Because of

$$|\Delta w_\varepsilon^b(x)| \leq C(|w_\varepsilon^b(x)| + 1)$$

we get immediately that $||w_\varepsilon^b||_{C^{1,\lambda}}$ has a uniform bound. To make further progress we must estimate the first derivatives of $I_\varepsilon^b(x)$ and $J_\varepsilon^b(x)$. As V has bounded derivatives to order three, $D\phi_a$ decays exponentially and uniformly and, as we have just seen, w_ε^b has bounded first derivatives, all bounds being independent of $\varepsilon < \varepsilon_1$ and b, we see that Δw_ε^b has its first derivatives bounded independently of $\varepsilon < \varepsilon_1$ and b. Thus we arrive at a uniform bound on $||w_\varepsilon^b||_{C^{2,\lambda}}$.

□

Note that under the stronger assumption that V is periodic, so that all its derivatives are bounded, we can obtain stronger results; for example the families u_ε^b and w_ε^b are then bounded in C^k for all k.

Theorem 15. *Let B satisfy the conditions of theorem 11. Then $u_\varepsilon^b\big(x + \frac{b}{\varepsilon} - s_\varepsilon^b\big)$ converges in $C^{2,\lambda}$ to ϕ_a as $\varepsilon \to 0$. The convergence is uniform with respect to b.*

Proof. This is an immediate consequence of theorem 14. □

Theorem 16. *Let B satisfy the conditions of theorem 11. Then $w_\varepsilon^b \to \eta^b$ in C^2 as $\varepsilon \to 0$, uniformly with respect to b.*

Proof. Let $\varepsilon_1 < \varepsilon_0$. The family w_ε^b, $0 < \varepsilon < \varepsilon_1$, $b \in B$ is bounded in $C^{2,\lambda}$. The same is true of the family η^b, $b \in B$. Hence it is also true of the family $w_\varepsilon^b - \eta^b$, $0 < \varepsilon < \varepsilon_1$, $b \in B$. It follows that the functions

$$D^\alpha(w_\varepsilon^b - \eta^b), \quad 0 < \varepsilon < \varepsilon_1, \quad b \in B, \quad |\alpha| \leq 2$$

form a uniformly equicontinuous family. If the claim of the theorem is false then we can find a multiindex α with $|\alpha| \leq 2$, sequences $\varepsilon_\nu \to 0$, $x_\nu \in \mathbb{R}^n$, $b_\nu \in B$ and a number $\delta > 0$ such that

$$|D^\alpha(w_{\varepsilon_\nu}^{b_\nu}(x_\nu) - \eta^{b_\nu}(x_\nu))| > \delta$$

By uniform equicontinuity there exists $r > 0$, independent of ν, such that

$$|D^\alpha(w_{\varepsilon_\nu}^{b_\nu}(y) - \eta^{b_\nu}(y))| > \frac{\delta}{2}$$

whenever $|y - x_\nu| < r$. But this contradicts the fact that $w_\varepsilon^b \to \eta^b$ in H^2 uniformly with respect to $b \in B$ as follows from theorem 11.

□

Theorem 17. *Let B satisfy the conditions of theorem 11 and let $0 < \varepsilon_1 < \varepsilon_0$. Then the family w_ε^b, $0 < \varepsilon < \varepsilon_1$, $b \in B$, decays uniformly at infinity.*

Proof. As shown above the family is relatively compact in H^2, hence also in L^2. It follows that the limit

$$\lim_{R\to\infty} \int_{|x|>R} (w_\varepsilon^b)^2 \, dx = 0$$

is attained uniformly with respect to ε and b. It is moreover a bounded family in C^2, hence uniformly equicontinuous, and individual members decay at infinity. The uniform decay now follows by the same kind of argument as we used for the uniform decay of ϕ_a. □

Theorem 18. *Let B satisfy the conditions of theorem 11 and let $\varepsilon_1 < \varepsilon_0$. Let $0 < \mu < \sqrt{\delta}$ (where δ is defined in the positivity assumption on F). Then there exists $C > 0$ independent of $\varepsilon < \varepsilon_1$ and $b \in B$, such that*

$$\left| u_\varepsilon^b\Big(x + \frac{b}{\varepsilon} - s_\varepsilon^b\Big) \right| < Ce^{-\mu|x|}.$$

Similar estimates hold for the derivatives of u_ε^b up to and including order 3.

Note: Obviously we may drop the vector s_ε^b from the estimate.

Proof. The function $v(x) := u_\varepsilon^b\big(x + \frac{b}{\varepsilon} - s_\varepsilon^b\big)$ satisfies a linear equation

$$-\Delta v + G_\varepsilon^b(x)v = 0$$

where

$$G_\varepsilon^b(x) = \int_0^1 \frac{\partial F}{\partial u}\Big(V(\varepsilon(x - s_\varepsilon^b) + b), t\phi_a(x) + t\varepsilon^2 w_\varepsilon^b(x)\Big)\, dx$$

We can write

$$\begin{aligned} G_\varepsilon^b(x) = \frac{\partial F}{\partial u}&(V(\varepsilon(x - s_\varepsilon^b) + b), 0) \\ &+ \phi_a(x) \int_0^1 \int_0^1 \frac{\partial^2 F}{\partial u^2}(V(\varepsilon(x - s_\varepsilon^b) + b), t_1 t_2 \phi_a(x)) t_1\, dt_1\, dt_2 \\ + \varepsilon^2 w_\varepsilon^b(x) &\int_0^1 \int_0^1 \frac{\partial^2 F}{\partial u^2}(V(\varepsilon(x - s_\varepsilon^b) + b), t_1\phi_a(x) + t_1 t_2 \varepsilon^2 w_\varepsilon^b(x)) t_1\, dt_1\, dt_2 \end{aligned} \tag{16}$$

The first term is higher than $\delta > 0$ independently of x, ε and b. The second and third decay at infinity uniformly with respect to $\varepsilon < \varepsilon_1$ and b by theorems 17 and theorem 4 and the boundedness lemmas. We therefore see that

$$\liminf_{|x|\to\infty} G_\varepsilon^b(x) \geq \delta$$

uniformly with respect to ε and b. Since we already know that $v(x)$ decays at infinity we see, by [5], that there exists a constant C independent of ε and b, such that

$$\left| u_\varepsilon^b\big(x + \frac{b}{\varepsilon} - s_\varepsilon^b\big) \right| \leq C||u_\varepsilon^b||_{L^\infty} e^{-\mu|x|}$$

Since we already have a uniform bound on $||u_\varepsilon^b||_{L^\infty}$ the theorem is proved. □

We conclude this section by proving the uniform positivity of u_ε^b given that $\phi_a > 0$.

Theorem 19. *Make the same assumptions as in theorem 11 and in addition assume that $\phi_a > 0$ for all $a \in I$. Then there exists $\varepsilon_1 \in]0, \varepsilon_0[$ such that $u_\varepsilon^b > 0$ for $0 < \varepsilon < \varepsilon_1$ and all $b \in B$.*

Proof. The function u_ε^b satisfies $-\Delta u_\varepsilon^b + F(V(\varepsilon x), u_\varepsilon^b) = 0$ which we write in the form

$$-\Delta u_\varepsilon^b + g_\varepsilon^b(x) u_\varepsilon^b = 0$$

where

$$g_\varepsilon^b(x) = \int_0^1 \frac{\partial F}{\partial u}(V(\varepsilon x), t u_\varepsilon^b)\, dt.$$

By theorem 15 we know that

$$u_\varepsilon^b\Big(\,\cdot + \frac{b}{\varepsilon} - s_\varepsilon^b\Big) \to \phi_{V(b)}$$

uniformly with respect to $b \in B$. We are assuming that $\phi_a > 0$. Hence there exists ε_1 such that if $0 < \varepsilon < \varepsilon_1$ and x is such that $u_\varepsilon^b(x) < 0$ then $g_\varepsilon^b(x) > \delta/2$ where δ was introduced in property (P1). Moreover ε_1 is independent of $b \in B$. From this we deduce that $u_\varepsilon^b > 0$ if $0 < \varepsilon < \varepsilon_1$ and $b \in B$, using exactly the same arguments as in [4, theorem 18].

□

7 Infinite bump solutions

Everything is now in place to give the two main conclusions of this paper. The proof of the first is a direct application of [4] using the regularity and uniformity properties of the last section. Note that these are not all required at the strength obtained; for example we only need to know that w_ε^b is uniformly bounded in C^1 (we have $C^{2,\lambda}$) and that $w_\varepsilon^b \to \eta^b$ in C^λ_{loc} uniformly with respect to b (it converges in the $C^2(\mathbb{R}^n)$-norm uniformly with respect to b). The application of [4] may require some strengthening of the regularity of V as we now see.

Theorem 20. *Make the same assumptions as in theorem 11, but assume in addition that the derivatives of V are bounded up to order* $\max(3, 2 + \frac{n}{2})$. *Let $0 < \lambda < 1$. Then there exists $\varepsilon_1 > 0$ such that if $0 < \varepsilon < \varepsilon_1$ and $B_0 \subset B$ there exists a unique solution $U_\varepsilon^{B_0}$ of the equation $-\Delta u + F(V(\varepsilon x), u) = 0$ such that*

$$\Big\| U_\varepsilon^{B_0} - \sum_{b \in B_0} u_\varepsilon^b \Big\|_{C^{2,\lambda}} \le C e^{-\sigma/\varepsilon}$$

where C and σ are positive constants independent of B_0 and ε.

Note that as was shown in [4] the series $\sum_{b\in B_0} u_\varepsilon^b$ converges in C^3_{loc} to a function in $C^3(\mathbb{R}^n)$.

The second main conclusion follows from [4] and theorem 19.

Theorem 21. *Under the same conditions as in theorem 20, if the solutions u_ε^b are strictly positive, so is the solution $U_\varepsilon^{B_0}$. In particular if the ground state ϕ_a is strictly positive for all $a \in I$ then there exists ε_2 such that $U_\varepsilon^{B_0} > 0$ whenever $0 < \varepsilon < \varepsilon_2$ and $B_0 \subset B$.*

8 Examples

The main difficulty in obtaining examples in dimensions $n \geq 2$ is the existence of the ground state with the required properties. There is clearly an abundance of functions V with infinitely many critical points that satisfy the required conditions and are not periodic. For example we can take V to be almost periodic. Or else V could be a periodic function multiplied by a bounded function, or the composition of a periodic function with a diffeomorphism of $\mathbb{R}^n$ onto itself. We shall explicitly exhibit some examples of the set B in cases when V is not periodic.

The ground state

Let p be an integer satisfying $1 < p < \frac{n+2}{n-2}$. Our starting point is the ground state solution $\psi(x)$ of

$$-\Delta u + u - u^p = 0$$

By ground state we mean here the non-trivial, radially symmetric solution of least energy. This is known to have all the required properties of exponential decay and quasi-non-degeneracy (see references [1], [7]). We generate the following two cases:

Case 1:

$$\phi_a(x) := a^{\frac{1}{p-1}} \psi(a^{\frac{1}{2}} x), \quad (a > 0)$$

is a solution of

$$-\Delta u + au - u^p = 0$$

Case 2:

$$\phi_a(x) := a^{\frac{1}{1-p}} \psi(x), \quad (a > 0)$$

is a solution of

$$-\Delta u + u - au^p = 0$$

It is clear that in both cases $\phi_a(x)$ satisfies the conditions (Φ1–4). We can now obtain conclusions for the two problems:

$$-\varepsilon^2 \Delta u + V(x)u - u^p = 0$$

$$-\varepsilon^2 \Delta u + u - V(x)u^p = 0$$

For the first problem the positivity condition requires V to have its range in the interval $]\delta, \infty[$, where $\delta > 0$. For the second problem the positivity condition is automatically satisfied but we must have $V(x) > 0$ as the ground state exists for $a > 0$.

In the one-dimensional case there is much more scope for constructing ground states with the required properties. We can study the problem

$$-\varepsilon^2 u'' + F(V(x), u) = 0$$

where, for a range of values of $a \in I$, we assume $F(a, 0) = 0$, $\frac{\partial F}{\partial u}(a, 0) > 0$, and $G(a, u) := -\int_0^u F(a, s)\, ds$ satisfies $\sup_{u>0} G(a, u) > 0$. Then the solution ϕ_a is a ground state where the curve $x \mapsto (\phi_a(x), \phi_a'(x))$ is the phase-plane trajectory in the region $u > 0$ that tends to the saddle point $(0, 0)$ as $|x| \to \infty$ and is such that ϕ_a is an even function.

More precisely let $u = r(a)$ be the lowest positive zero of $G(a, u)$. Then ϕ_a is the solution of $-u'' + F(a, u) = 0$ that satisfies the Cauchy conditions $u(0) = r(a)$, $u'(0) = 0$. If we make the natural assumption that $r(a)$ is a smooth function of $a \in I$ then $\phi_a(x)$ will be a smooth function of a. Noting that ϕ_a satisfies $\frac{1}{2}\phi_a'(x)^2 + G(a, \phi_a(x)) = 0$ and using the assumption that $\frac{\partial F}{\partial u}(a, 0) > 0$ one can easily deduce from this the exponential decay of ϕ_a and its derivatives at $x = \pm\infty$ and that the decay is uniform with respect to a if the latter is restricted to a compact subset of I. This also gives the continuity of $a \mapsto \phi_a$ from I to H^2. That ϕ_a is quasi-non-degenerate follows by observing that the Wronskian of a solution base of $-v'' + \frac{\partial F}{\partial u}(a, \phi_a(x))v = 0$ is a non-zero constant so that one cannot have two linearly independent solutions that decay at $\pm\infty$.

It is interesting to study the condition (Φ4) in this context. A short calculation shows that (Φ4) is equivalent to

$$\int_0^{r(a)} |G(a, u)|^{-\frac{1}{2}} \frac{\partial G}{\partial a}(a, u)\, du \neq 0.$$

This is, for example, satisfied if $\frac{\partial G}{\partial a}(a, u)$ is of one sign in the interval $]0, r(a)[$.

Instances of the set B

We shall give some explicit examples of the set B in non-periodic cases based on some quite elementary calculations. For definiteness consider the problem

$$-\varepsilon^2 u'' + u - V(x)u^3 = 0, \quad -\infty < x < \infty$$

where the existence of the ground state requires $V(x) > 0$.

We look first at a couple of cases where $V(x)$ is almost periodic. As a first example let

$$V(x) = 2 + \sin x + \sin \sqrt{10}x.$$

We can let B consist of the set of all critical points. They are the solutions of

$$\cos x + \sqrt{10} \cos \sqrt{10}x = 0$$

which is obviously an infinite set. Now we have for $x \in B$

$$|V''(x)| = \left|\sin x + 10 \sin \sqrt{10}x\right| = \left|\sin x \pm 10\sqrt{1 - \frac{\cos^2 x}{10}}\right| > 8.$$

We modify the first example and let

$$V(x) = 2 + \sin x + \frac{1}{\sqrt{10}} \sin \sqrt{10}x.$$

The critical points of V consist of the two arithmetic progressions

$$\frac{(2m+1)\pi}{\sqrt{10}+1}, \quad \frac{(2n+1)\pi}{\sqrt{10}-1}, \quad m, n \in \mathbb{Z}$$

These are disjoint from each other and do not together form an arithmetic progression. However in this case we cannot choose for B the set of all critical points; it is clear that the distance between consecutive critical points has infimum 0. A short calculation shows that in order to satisfy condition (2) of theorem 11 we want to choose infinite subsets $M, N \subset \mathbb{Z}$ such that the sets

$$\frac{(2m+1)}{\sqrt{10}+1}, \quad \frac{(2n+1)}{\sqrt{10}-1}, \quad m \in M, \quad n \in N$$

are at a positive distance from the set $\mathbb{Z}$. Then we may take

$$B = \left\{\frac{(2m+1)\pi}{\sqrt{10}+1} : m \in M\right\} \cup \left\{\frac{(2n+1)\pi}{\sqrt{10}-1} : n \in N\right\}$$

To construct M we proceed as follows. Corresponding to each open interval $k(\sqrt{10}+1) < x < (k+1)(\sqrt{10}+1)$ place in M the integer j where $2j+1$ is the odd integer in the interval lying furthest from the endpoints. For N we proceed similarly using the intervals $k(\sqrt{10}-1) < x < (k+1)(\sqrt{10}-1)$. We used $\sqrt{10}$ here to ensure that each of both batches of intervals would always contain at least one odd integer. Obviously any irrational number λ could replace $\sqrt{10}$ but this simple prescription for B works when $\lambda^2 > 3$.

Our third example is

$$V(x) = e^{G(x)} \cos x + C$$

where $G(x)$ has all derivatives bounded, and G' and G'' are small enough. We take for B the set of all critical points. They are roots of

$$\tan x = G'(x)$$

which are clearly infinitely many. Now for $x \in B$ we have

$$V''(x) = e^{G(x)} \cos x (G''(x) - G'(x)^2 - 1).$$

We therefore obtain an inequality $|V''(x)| > \delta > 0$ for all $x \in B$ if, for example, $|G''(x) - G'(x)^2| < \alpha < 1$ for all x. Note that $\cos x$ stays away from 0 because $\tan x$ is bounded above in B.

Another source of oscillating but non-periodic functions arises from solving second order linear differential equations. For example let $V(x) = y(x) + C$ where $y(x)$ is a non-trivial solution of

$$y'' + \left(1 + \frac{1}{1+x^2}\right) y = 0$$

and C is chosen to make $V(x)$ positive. The solutions are oscillating (have infinitely many zeros and critical points) by Sturmian theory and are bounded, in fact they approach solutions of $y'' + y = 0$ at ∞ and $-\infty$. The latter follows by standard arguments using the fact that $1/(1+x^2)$ is integrable at infinity. Since the Wronskian of two solutions is a constant it is easy to check that $y''(x)$ stays away from 0 at critical points of $y(x)$ so that we can take as B the set of all critical points.

A more explicit example of this kind is $V(x) = xj_p(x) + C$ where $j_p(x)$ is the spherical Bessel function with order equal to the non-negative integer p. The function $y(x) = xj_p(x)$ satisfies the differential equation

$$y'' + \left(1 - \frac{p(p+1)}{x^2}\right) y = 0$$

and up to a constant multiplier it is the unique solution that extends to an entire analytic function. As in the last example, and by the same argument, we can take as B the set of all critical points, with the exclusion of $x = 0$ which is degenerate if $p \geq 1$.

An example of a quite different kind is

$$V(x) = 1 + \cos\left(x + \frac{1}{x^2+1}\right).$$

Now the function $y(x) = x + \frac{1}{x^2+1}$ is a diffeomorphism of $\mathbb{R}$ onto itself and $|y'(x)| > \frac{1}{2}$ (rather comfortably). Hence the critical points are the roots of

$$\sin\left(x + \frac{1}{x^2+1}\right) = 0$$

which is to say they are the points $b_n = y^{-1}(n\pi)$, $n \in \mathbb{Z}$; quite easy to compute numerically. Moreover

$$|V''(b_n)| = |y'(b_n)| > \frac{1}{2}$$

so that we may take as B the set of all critical points b_n.

This example generalizes easily to n dimensions as follows. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a periodic function with critical point set C, all members of which are non-degenerate. We suppose that there are only finitely many critical points in each cell of the period lattice of f. Now consider

$$V(x) = f(y(x))$$

where $y : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism with derivatives bounded to a suitable order and such that $|Dy(x)| > \delta > 0$ for all $x \in \mathbb{R}^n$. Then we may take

$$B = y^{-1}(C).$$

References

[1] M. K. Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. Arch. Rational Mech. Anal. 105, 243–266 (1989).

[2] R. Magnus. A scaling approach to bumps and multi-bumps for nonlinear partial differential equations. Proceedings of the Royal Society of Edinburgh 136A, 585-614 (2006).

[3] R. Magnus and O. Moschetta. Non-degeneracy of perturbed solutions of semilinear partial differential equations. Ann. Acad. Sci. Fenn. Math. 35, 75–86, (2010).

[4] R. Magnus and O. Moschetta. The non-linear Schrödinger equation: non-degeneracy and infinite-bump solutions. Preprint, University of Iceland.

[5] C. A. Stuart and P. Rabier. Exponential decay of the solutions of quasilinear second-order equations and Pohozaev identities. Journal of Differential Equations 165, 199–234 (2000).

[6] X. Wang. On concentration of positive bound states of nonlinear Schrdinger equations. Comm. Math. Phys. 153, 229-244 (1993).

[7] M. Weinstein. Modulational stability of the ground states of nonlinear Schrdinger equations. SIAM J. Math. Anal. 16, 567-576 (1985).