

Thesis for the degree of Magister Scientiarum in Physics

Semi-Classical Charged Black Holes

September 16, 2010

Friðrik Freyr Gautason



UNIVERSITY OF ICELAND

School of Engineering and Natural Sciences

Faculty of Physical Sciences

Reykjavík, September 2010

Semi-Classical Charged Black Holes

A 60 credit thesis submitted in partial fulfillment of the requirements for the degree of Magister Scientiarum in Physics at the University of Iceland.

Committee in charge

Prof. Lárus Thorlacius, advisor

NORDITA, Nordic Institute for Theoretical Physics

Prof. Þórður Jónsson

University of Iceland

Moderator

Gunnlaugur Björnsson

Science Institute – University of Iceland

Semi-Classical Charged Black Holes

© 2010 Friðrik Freyr Gautason

Science Institute Report: RH-17-2010

Contents

Abstract	iv
Ágrip (in Icelandic)	v
Acknowledgements	vi
1 Introduction	1
1.1 Mass Inflation	1
1.2 Pair Creation of Charged Particles	2
2 Two Dimensional Models	5
2.1 Equations of Motion	7
2.2 Coordinates	8
2.3 Vacuum Solution	9
2.4 Massive Scalar Field	10
2.5 Quantum Corrections	11
3 Numerical Black Hole Solutions	14
3.1 The Dynamical Problem	14
3.2 The Numerical Solver	16
3.3 Results	19
3.4 Large Black Hole	21
4 Conclusions	25

Abstract

In this thesis we take a look at charged black holes. We introduce a 1+1 dimensional dilaton gravity model coupled to electric field. A static solution to this model has the same causal structure as the Reissner-Nordström solution. We introduce classical charged scalar matter to study the formation of charged black holes from dynamical collapse. We observe signals of the Cauchy horizon as well as inflation of the mass parameter as expected. When quantum effects are introduced the evolution is modified. The charge evaporates and the signal of the Cauchy horizon disappears.

Ágrip (In Icelandic)

Í þessari ritgerð ransökum við innri gerð hlaðinna svarthola. Til þess innleiðum við $1+1$ víða þyngdarfræðikenningu sem víxlverkar við rafsvið. Tímaóháð lausn þessa líkans hefur svipaða eiginleika og Reissner-Nordström lausnin. Við innleiðum klassískt hlaðið skalarsvið til þess að skoða myndun hlaðinna svarthola úr tómarúmi. Eins og búast má við sjáum við merki um Cauchy sjóndeild sem og massaþan. Þegar skammtahrif eru tekin með í reikninginn verður tímaþróunin öðruvísi. Hleðsla svartholsins gufar upp og merki um Cauchy sjóndeildina hverfa.

Acknowledgements

First of all I would like to thank my advisor Lárus Thorlacius for his guidance during my Masters studies. Even though we spent much time in different countries he has taught me well and given me confidence in my work. I also would like to thank Þórður Jónsson for helpful discussions and Jan Valdman for his friendly assistance with numerics.

My fellow graduate students in mathematics and physics deserve thanks for many interesting discussions and joyful moments at the office.

This work was done at the Science Institute – University of Iceland and was supported by grants from the University of Iceland Research Fund and Icelandic Research Fund.

I am very grateful for the hospitality I received during my visits to the Nordic Institute for Theoretical Physics, NORDITA in the winter of 2008-2009 and again in January of 2010.

Finally I would like to thank my family, especially my wife María for her endless love and support and my son Hallmar Gauti who has made me very proud for the last three months.

1

Introduction

In this thesis we will study charged black holes and the effects quantum corrections have on their global geometry. The Reissner-Nordström solution is a static spherically symmetric black hole solution describing a charged black hole [1]. It is a two parameter solution where the parameters represent the mass and the charge of the black hole. We will be interested in the case where the mass is greater than the charge in natural units. In that case the solution has a time-like singularity hidden behind a pair of horizons, an outer and an inner horizon. The presence of a time-like singularity indicates that the inner horizon is a Cauchy horizon through which dynamic evolution can not continue. It also means that an observer who enters the black hole can escape the singularity into an asymptotically flat region. This region is different from the one he started in, and the solution can be extended in such a way to contain infinite number of asymptotically flat regions. The Penrose diagram for the maximally extended Reissner-Nordström solution can be observed in figure 1.1.

1.1 Mass Inflation

The time-like singularity clearly imposes some problems which must be addressed. First of all the inability to evolve physics beyond the Cauchy horizon is very unsatisfactory. We would have to impose boundary conditions along the singularity and thereby making assumptions about Planck scale physics. Second, the black hole complementarity solution to the information paradox also loses its value if observers are allowed to escape the singularity into a different asymptotically flat region [2]. Finally, it also turns out that the inner horizon is a surface of infinite blue-shift and

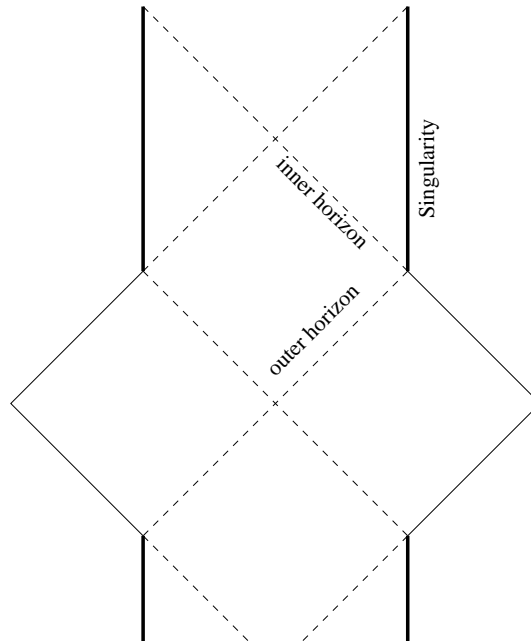


Figure 1.1: The Penrose diagram for a maximally extended Reissner-Nordström black hole with mass greater than charge. The diagram should be repeated infinite number of times in both vertical directions.

is thus unstable [3–5]. Poisson and Israel [6] further showed that if ingoing and outgoing energy fluxes meet at the inner horizon the mass parameter of the black hole grows exponentially. This results in infinite curvature at the inner horizon which means that the horizon is replaced by a null singularity which turns space-like inside the black hole. Ori [7] has showed that the null singularity might be penetrable by a free falling observer and is therefore a weak singularity [8]. The effect discovered by Poisson and Israel is called mass inflation and has been confirmed in numerical simulation of formation of black hole from collapsing spherical shells of charged matter [9]. Figure 1.2 shows a Penrose diagram which represents the solutions which have been found in these numerical simulations.

1.2 Pair Creation of Charged Particles

Hawking showed that quantum corrections have great effect on the evolution of black holes [10]. Our picture of charged black holes might not be complete without introducing some quantum effects, and so the figure 1.2 could be inaccurate. All the problems mentioned above are associated with the fact that the black hole is charged and it is in order to look at quantum corrections that affect the charge

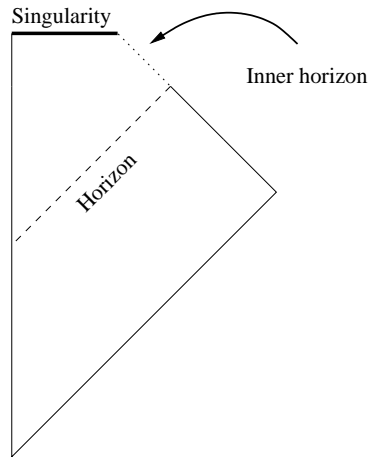


Figure 1.2: The Penrose diagram for a Charged black hole formed by spherical shells of charged matter. The inner horizon is present at infinite advanced time.

of the hole. Strong electric fields drive the pair creation of charged particles [11]. This effect is called Schwinger effect after its discoverer, but Schwinger was able to calculate the production rate of charged particles in a constant electric field. In the vicinity of the outer horizon of a black hole this pair creation results in evaporation of the black hole charge, but it is not clear how pair creation inside the black hole affects the evolution of the black hole geometry.

Several others tried to use this known rate to look at the effect it has on black hole geometry. Early analytic work was carried out by Herman and Hiscock [12] who made rather crude approximations and found that the evaporation of charge can result in the elimination of the Cauchy horizon. Later Sorkin and Piran [13] did some numerical analysis on a simplified model based on Schwinger's work but they failed to take in account the back-reaction of the evaporating particles on the geometry. Their result was that Schwinger effect does not eliminate the Cauchy horizon.

In this thesis we will build on work of Frolov, Kristjansson and Thorlacius [14,15]. We use the fermion boson duality in two dimensions to model the quantum effects on the geometry, this means using a two dimensional model to simulate the four dimensional gravity and then using well known results in two dimensions. In a straight follow up on the work carried out by Frolov, Kristjansson and Thorlacius we perform numerical simulations of formation and evolution of the charged black hole. They did some preliminary analysis of the model but the resolution of the numerical data was not sufficient to state accurately if the inner horizon had formed. We improve on their work by increasing the resolution, using a new method based on

the conformal symmetry of the model, Which allows us to concentrate the numerical computation on the black hole space-time which is of interest for the problem at hand.

2

Two Dimensional Models

To be able to study the phenomena discussed above we introduce a simplified 1+1 dimensional model. It is based on a much researched model proposed by Callan, Giddings, Harvey and Strominger (CGHS) [16]. The advantage of working in only two dimensions is that known results can be applied when considering quantum effects which is our final goal. The CGHS model has a static black hole solution which has the same qualitative behaviour as the four dimensional Schwarzschild solution. When electric field is added, the model has a static charged black hole solution which is analogous to the Reissner-Nordström solution as we will see later on.

The action for the CGHS model can be derived from a four dimensional model although we will not be interested in its origin. We will merely think of it as a good toy model for the more complicated four dimensional gravity. The action reads

$$S_G = \int \sqrt{-g} e^{-2\phi} \{R + 4(\nabla\phi)^2 + 4\lambda^2\} d^2x. \quad (2.1)$$

where g is the determinant of the two dimensional metric. The scalar curvature of the space-time is denoted by R and ϕ is called the dilaton. We will consider ϕ to be a scalar field which enables black hole solutions in otherwise trivial two-dimensional gravity. Note also that the coupling strength is governed by the quantity $\psi = e^{-2\phi}$ which we call the area function, a term inherited from the four dimensional theory. Since we are interested in charged black holes we need to add a $U(1)$ gauge field to the above action. It turns out that the correct way is to add

$$S_F = -\frac{1}{4} \int \sqrt{-g} e^{-2\phi} F^2 d^2x$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and $F^2 = F_{\mu\nu}F^{\mu\nu}$. The total action will be $S = S_G + S_F + S_M$ where S_M is an arbitrary matter action which does not involve the dilaton ϕ .

The CGHS action can be compared with a spherical reduction of the four dimensional Einstein gravity. We write the 1+3 dimensional line element as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + r^2 d\Omega^2$$

where $d\Omega^2$ is the line element of the transverse two-sphere and $g_{\mu\nu}$ is the metric of the two-dimensional sub-manifold. By inserting this into the full Einstein-Hilbert action we can simplify to get the effective two dimensional action

$$S_{\text{SR}} = 4\pi \int \sqrt{-g}r^2 \left(R - 2\frac{\nabla^2 r}{r} + \frac{2}{r^2} + F^2 \right) d^2x.$$

Here $F^2 = F_{\mu\nu}F^{\mu\nu}$ in two dimensions as before but the corresponding four dimensional vector field $A_\mu^{(4)}$ must be spherically symmetric and its components in the spherical directions must be zero. To this we could add a two dimensional matter action S_M if we can motivate the form of S_M by integrating out the spherical directions, using spherical symmetry of the matter fields. In general the matter action takes the form

$$S_M = \int r^2 \mathcal{L}_M d^2x$$

where \mathcal{L}_M is the Lagrange density of the corresponding two dimensional fields. Adding matter to the CGHS model is simpler which makes inclusion of quantum effects consistent. Note that r is the radius of the transverse two-sphere and we can think of it as a scalar function which depends on the two coordinates that have not been integrated out. By defining $e^{-2\phi} = 4\pi r^2 \lambda^2$ in analogy with the CGHS theory, where λ is just some constant to make ϕ dimensionless, we can write the action

$$S_{\text{SR}} = \frac{1}{\lambda^2} \int \sqrt{-g}e^{-2\phi} \left(R + 2(\nabla\phi)^2 + 8\pi\lambda^2 e^{2\phi} - \frac{1}{4}F^2 \right) d^2x.$$

Hence this is similar to the CGHS model. As we have mentioned the CGHS model has the advantage over the above reduced model, that quantum effects can be included in consistent manner.

2.1 Equations of Motion

Now we return to the CGHS model and consider the action $S = S_G + S_F + S_M$ where S_M is unspecified matter action. We calculate the equations of motion by varying the action with respect to the metric and the dilaton. First note the two results

$$\delta R = (R_{\mu\nu} + g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu)\delta g^{\mu\nu}$$

and

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.$$

Using this and that $\delta S = 0$ we get the Einstein equation

$$\begin{aligned} \frac{\delta S}{\delta g^{\mu\nu}} = & \sqrt{-g} \left\{ 4e^{-2\phi}\nabla_\mu\phi\nabla_\nu\phi - \nabla_\mu\nabla_\nu e^{-2\phi} - \frac{1}{2}F_{\mu\alpha}F_\nu{}^\alpha e^{-2\phi} \right. \\ & \left. + g_{\mu\nu} \left(\nabla^2 e^{-2\phi} - \frac{e^{-2\phi}}{8}(16(\nabla\phi)^2 + 16\lambda^2 - F^2) \right) \right\} + \frac{\delta S_M}{\delta g^{\mu\nu}} = 0, \end{aligned}$$

where the last term is nothing but the energy-momentum tensor

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}.$$

Taking the trace of the Einstein equation gives

$$\nabla^2 e^{-2\phi} - \frac{e^{-2\phi}}{4}(16\lambda^2 + F^2) = T \quad (2.2)$$

where $T = T^\mu{}_\mu$ is the trace of the energy-momentum tensor. Now varying with respect to the dilaton gives us

$$\frac{\delta S}{\delta\phi} = -2\sqrt{-g}e^{-2\phi} \left(R + 4\nabla^2\phi - 4(\nabla\phi)^2 + 4\lambda^2 - \frac{1}{4}F^2 \right) = 0,$$

We can use the trace equation (2.2) to simplify the above and get

$$R + 2\nabla^2\phi = \frac{1}{2}F^2 + e^{2\phi}T. \quad (2.3)$$

Finally we get the Maxwell equation by varying the action with respect to the vector field A_μ ,

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta A_\mu} = \nabla_\nu (e^{-2\phi}F^{\nu\mu}) - J^\mu = 0, \quad (2.4)$$

where

$$J^\mu = -\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta A_\mu}$$

is the $U(1)$ current.

2.2 Coordinates

We will work exclusively in light cone coordinates which we label y^+ and y^- . We then choose a conformal gauge for the metric

$$[g_{\mu\nu}] = -\frac{e^{2\rho}}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where ρ is the conformal factor. Note that we still have some gauge freedom since we can make conformal transformation on the coordinates. That means if $x^+ = x^+(y^+)$ and $x^- = x^-(y^-)$ we have

$$\rho(x^+, x^-) = \rho(y^+, y^-) - \frac{1}{2} \log \frac{dx^+ dx^-}{dy^+ dy^-}.$$

In the conformal gauge the non-vanishing Christoffel symbols are

$$\Gamma_{++}^+ = 2\partial_+\rho \quad \text{and} \quad \Gamma_{--}^- = 2\partial_-\rho.$$

The square-root of the determinant of the metric and the curvature scalar are given by

$$\sqrt{-g} = \frac{e^{2\rho}}{2} \quad \text{and} \quad R = 8e^{-2\rho}\partial_+\partial_-\rho.$$

Finally for any scalar we have the simple identity

$$(\nabla f)^2 = g^{\mu\nu}(\nabla_\mu f)(\nabla_\nu f) = -4e^{-2\rho}\partial_+f\partial_-f, \quad \nabla^2 f = -4e^{-2\rho}\partial_+\partial_-f.$$

We can now rewrite our equations (2.2) and (2.3) in these coordinates

$$\left. \begin{aligned} \square\psi &= \frac{e^\sigma}{4\psi} (16\lambda^2\psi + F^2\psi + 4T) \\ \square\sigma &= -\frac{e^\sigma}{2\psi^2} (F^2\psi + 2T) \end{aligned} \right\} \quad (2.5)$$

where we have defined $\psi = e^{-2\phi}$ and $\sigma = 2(\rho - \phi)$. We also use $\square = -4\partial_+\partial_-$ for the flat-space Laplacian. We have two more equations resulting from putting

diagonal components of the metric to zero. That is the $\pm\pm$ components of the original Einstein equation derived in last section

$$-\partial_{\pm}^2\psi + \partial_{\pm}\sigma\partial_{\pm}\psi = T_{\pm\pm}. \quad (2.6)$$

These are constraint equations and not independent of the two wave equations. This means that if the boundary conditions used to solve the wave equations satisfy the constraints, the solutions to the wave equations will automatically satisfy the constraints.

The non-zero components of the Maxwell field strength tensor are $F_{+-} = -F_{-+}$, so we find $F^2 = -8\psi^2 e^{-2\sigma}(F_{+-})^2$. In the conformal gauge the Maxwell equation (2.4) takes the form

$$\partial_{\pm} \left(\frac{\psi^2 F_{+-}}{e^{\sigma}} \right) = \pm \frac{1}{2} J_{\pm}.$$

2.3 Vacuum Solution

Lets look more closely at the equations when we take $S_M = 0$, that is $T_{\mu\nu} = J_{\mu} = 0$. If we also let $F_{\mu\nu} = 0$ the equations are those of the original CGHS model. The second of equations (2.5) then reads $\square\sigma = 0$ which integrates to $\sigma = f^+(y^+) + f^-(y^-)$ where f^{\pm} are arbitrary functions. We can choose coordinates (x^+, x^-) such that $\sigma(x^+, x^-) = 0$ using the transformation property discussed above. Then the other equation reads $\square\psi = 4\lambda^2$. This can be directly integrated by using the constraints to give

$$\psi = -\lambda^2 x^+ x^- + M, \quad (2.7)$$

where M is an integration constant interpreted as the mass of a black hole. Other integration constants are non-physical and have been put to zero. The coordinates chosen such that $\sigma = 0$ are called Kruskal coordinates and the solution (2.7) is the static black hole solution which we will use in what follows, then we will put $M = 0$ and refer to (2.7) as the vacuum solution. The curvature scalar can be calculated for this solution

$$R = \frac{4\lambda^2 M}{M - \lambda^2 x^+ x^-}$$

and clearly blows up when $M = \lambda^2 x^+ x^-$ except if $M = 0$. When $M < 0$ the singularity is time-like and unphysical. However when $M > 0$ we get a similar geometry as for the four dimensional Schwarzschild geometry, in particular the causal structure is the same [1].

If we on the other hand let $F_{\mu\nu} \neq 0$ then we can get a Reissner-Nordström like solution which we will describe but not derive, the details can be found in [17]. This is a static solution written out in (t, x) coordinates such that $\phi = -x$. The line element reads

$$ds^2 = -b(x)dt^2 + \frac{dx^2}{b(x)}$$

where $b(x) = 1 - M/\psi + Q^2/(8\psi^2)$. The field strength tensor has the form $F_{tx} = Q/\psi$. This is a two parameter family of solutions where M is interpreted as the mass of the black hole and Q is its charge. This solution has all the same qualitative features as the Reissner-Nordström solution, it has two horizons which are zeros of the function $a(x)$ as long as $M > |Q|/\sqrt{2}$ which is the case we are interested in.

Later we will consider dynamic collapse of charged matter that forms a charged black hole. Then the charge and the mass of the black hole will not be constants. We will do this in the conformal gauge and so we have to be able to calculate the charge and the mass as a function of the fields in these coordinates. The charge is simply given as the field strength tensor $Q = \psi^2 F_{+-}/e^\sigma$ and $\partial_\pm Q = \pm J_\pm/2$. The quantity Q measures the charge inside the point in question for fixed time. Now, to find the mass for a given charge we calculate

$$\frac{1}{b(x)} = g_{xx} = g(\partial_x, \partial_x) = g\left(\frac{\partial x^\mu}{\partial x} \partial_\mu, \frac{\partial x^\nu}{\partial x} \partial_\nu\right) = 2g_{+-} \frac{1}{\partial_+ \phi \partial_- \phi}$$

using that $x = -\phi$. We can further calculate

$$M = \psi \left(1 + \frac{e^{-\sigma}}{4\psi} \partial_+ \psi \partial_- \psi + \frac{Q^2}{8\psi^2} \right). \quad (2.8)$$

We use this formula to confirm the mass inflation close to the inner horizon when classical matter dynamically forms a charged black hole.

2.4 Massive Scalar Field

We are mainly interested in two matter actions, first is that of complex scalar field coupled to the $U(1)$ gauge field. This we will only consider at the classical level. Later we are interested in quantum effects and so we must modify our equations. The action for the classical scalar field is

$$S_M = -\frac{1}{2} \int d^2x \sqrt{-g} \{ |Df|^2 + m^2 |f|^2 \}$$

where $D_\mu f = \nabla_\mu f - ieA_\mu f$ is the $U(1)$ covariant derivative. This gives us the energy momentum tensor

$$T_{\mu\nu} = \frac{1}{4} \{ 2 \operatorname{Re} [(D_\mu f)^* (D_\nu f)] - g_{\mu\nu} (|Df|^2 + m^2 |f|^2) \}$$

which means that $T = -\frac{1}{2}m^2|f|^2$ and $T_{\pm\pm} = \frac{1}{2}|D_\pm f|^2$. The resulting equations are

$$\left. \begin{aligned} \square\psi &= \frac{e^\sigma}{4\psi} (16\lambda^2\psi - 2m^2|f|^2 + \psi F^2) \\ \square\sigma &= \frac{e^\sigma}{2\psi^2} (m^2|f|^2 - \psi F^2). \end{aligned} \right\} \quad (2.9)$$

The electromagnetic current for this matter field is

$$J_\mu = -e \operatorname{Im} [f^* D_\mu f].$$

By fixing the gauge freedom for the electromagnetic potential so that $A_+ = 0$ we get a wave equation for $a = A_-$

$$\square a = -2e \frac{e^\sigma}{\psi^2} \operatorname{Im} [f^* (\partial_- - iea)f] + 4\partial_+ a \partial_- (2 \log \psi - \sigma) \quad (2.10)$$

and a constraint equation

$$\partial_+ \left(\frac{\psi^2}{e^\sigma} \partial_+ a \right) = -\frac{1}{2} e \operatorname{Im} [f^* \partial_+ f]. \quad (2.11)$$

Finally we of course get a wave equation for the complex scalar f

$$D^2 f - m^2 f = 0,$$

which in conformal coordinates reads

$$\square f = -2ie (\partial_+ (af) + a\partial_+ f) + \frac{m^2 e^\sigma}{\psi} f. \quad (2.12)$$

2.5 Quantum Corrections

The above equations for the scalar field will mainly be used to test our model and also the numerics. Our main interest is to see the effect of pair creation of charged particles on the formation of inner horizon and how this contrasts with the mass inflation scenario which we expect to see numerically for the classical scalar field. To

be able to study the pair creation of charged particles we add to our original action (2.1) the standard action for a charged fermion Ψ . The reason we add fermions rather than scalar field is that in two dimensions the quantum electrodynamics of massive fermions is equivalent to scalar field theory with a non-linear sine-Gordon term as explained in [14] and references therein. For reference we write down the action for the charged fermion

$$S_M = \int \sqrt{-g} \{ i\bar{\Psi}\gamma^\mu \mathcal{D}_\mu \Psi - m\bar{\Psi}\Psi \} d^2x,$$

where Ψ is the fermion field γ^μ are the Dirac matrices and \mathcal{D}_μ is the spinor covariant derivative involving the $U(1)$ connection. That is $\mathcal{D}_\mu \Psi = (\partial_\mu + \frac{i}{2} J_{ab} \omega_{\mu}^{ab} + ieA_\mu) \Psi$. Let Z be the bosonized fermion field, then the equivalent action to the one above is given by

$$S_M = - \int d^2x \left\{ \sqrt{-g} \left[\frac{1}{2} (\nabla Z)^2 + V(Z) \right] + \frac{e}{\sqrt{4\pi}} \epsilon^{\mu\nu} F_{\mu\nu} Z \right\}$$

where

$$V(Z) = em \left[1 - \cos \left(\sqrt{4\pi} Z \right) \right].$$

Note that here we have absorbed an unimportant numerical constant into the mass m . The fermion-boson duality we use is a strong-weak duality which means that the classical scalar action is equivalent to a strongly coupled fermion system as long as $m \ll e$. This is precisely the reason we use the duality since real electrons have large charge to mass ratio. The Levi-Civita density $\epsilon^{\mu\nu}$ appears in the coupling to the $U(1)$ field and in the conformal coordinates has the value $\epsilon^{-+} = 1$. The $U(1)$ current can be calculated as before to give

$$J^\mu = \frac{e}{\sqrt{\pi}} \nabla_\nu (\epsilon^{\mu\nu} Z),$$

where we have defined $\varepsilon = e/\sqrt{-g}$. That means that the Maxwell equation will be almost trivial to solve

$$\nabla_\nu \left(\psi F^{\nu\mu} + \frac{e}{\sqrt{\pi}} \varepsilon^{\nu\mu} Z \right) = 0.$$

We write the solution as

$$F^{\nu\mu} = \frac{1}{\psi} \frac{e}{\sqrt{\pi}} (q - Z) \varepsilon^{\nu\mu}, \quad (2.13)$$

where we use $eq/\sqrt{\pi}$ as the integration constant. Note that the charge is now given by

$$Q = \frac{1}{2} \frac{e}{\sqrt{\pi}} (q - Z).$$

We can now replace $F_{\mu\nu}$ by the expression above in the remaining equations. The energy-momentum tensor is given by

$$T_{\mu\nu} = \frac{1}{4} \{ 2\nabla_\mu Z \nabla_\nu Z - g_{\mu\nu} ((\nabla Z)^2 + 2V(Z)) \},$$

which means that

$$T = -V(Z) \quad \text{and} \quad T_{\pm\pm} = \frac{1}{2} (\nabla_\pm Z)^2.$$

We can rewrite our wave equations (2.5) by using the above as well as an expression for F^2 in terms of Z to give

$$\left. \begin{aligned} \square\psi &= \frac{e^\sigma}{2\pi} \left(8\lambda^2\psi - \frac{1}{\psi} \frac{e^2}{\pi} (q - Z)^2 - 2V(Z) \right), \\ \square\sigma &= \frac{e^\sigma}{\psi^2} \left(\frac{1}{\psi} \frac{e^2}{\pi} (q - Z)^2 + V(Z) \right). \end{aligned} \right\} \quad (2.14)$$

Finally varying the matter action with respect to Z gives us the wave equation

$$\square Z = \frac{e^\sigma}{\psi} \left(V'(Z) - \frac{1}{\psi} \frac{e^2}{\pi} (q - Z) \right). \quad (2.15)$$

3

Numerical Black Hole Solutions

The main subject of this thesis is to explore the formation of inner horizon in dynamic processes. We do this by solving the equations in chapter 2 numerically. We have to focus our calculations close to the black hole to be able to discover the presence of the inner horizon. This of course can be achieved by adaptive methods but it is complicated and slow for the algorithm described in section 3.2. Instead we make use of the conformal symmetry of the equations and use coordinate transformations to focus the computer time in the appropriate region.

3.1 The Dynamical Problem

Take Ω to be a rectangular domain in some conformal coordinate chart in which we want to find a solution to the equations (2.9) or (2.14). The coordinates are labeled by (y^+, y^-) . Let Σ^+ be the lowermost boundary of Ω and Σ^- be the leftmost boundary of Ω , see figure 3.1. We want to form a black hole from vacuum and so the fields ψ and σ must take their vacuum values at Σ^- . All other fields (f, a and Z) must be zero at this boundary. On the Σ^+ boundary however we want the matter fields to enter, though still away from Σ^- . To be accurate the matter fields should enter from past infinity, however if we make sure that ψ is large on the Σ^+ boundary the coupling of the matter fields is negligible. However since the electric field is always strongly coupled we must take the initial charge to be zero. For the classical model we choose the form of f and from there solve the constraint equation (2.11) to find a . For the quantum corrected model we just choose the value of Z . To find σ and ψ on Σ^+ we choose σ motivated by the vacuum solution and solve the constraint equations to get the value of ψ . Now that all initial conditions have been

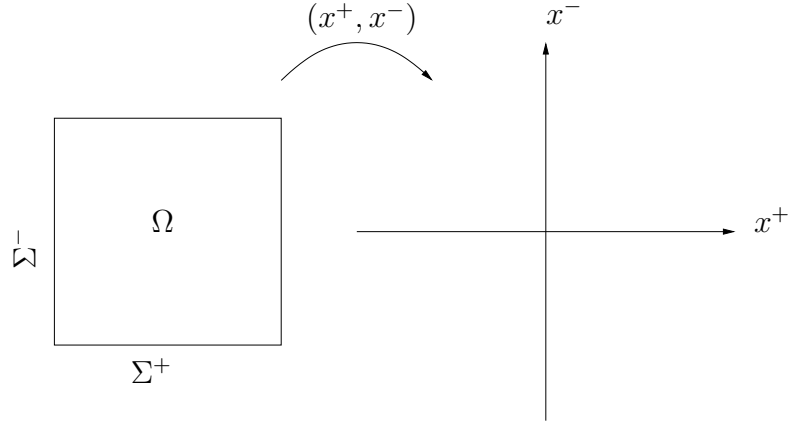


Figure 3.1: On the left hand side we see a simple box coordinate patch where the coordinates are labeled by (y^+, y^-) , these are related to the Kruskal coordinates on the right via the conformal mappings $(x^+(y^+), x^-(y^-))$.

specified and we are sure that they satisfy all constraint equations involved we can use the corresponding wave equations to find the values of the fields in all of Ω .

The numerical solver assumes a simple form for the domain Ω , that is $\Omega = [0, 1] \times [0, 1]$ in the (y^+, y^-) coordinate chart. However the relation to the Kruskal coordinates (x^+, x^-) must be specified before starting the calculations. Effectively we choose the functions $x^+(y^+)$ and $x^-(y^-)$. Then the vacuum value of σ can be calculated

$$\sigma(y^+, y^-) = \log \frac{dx^+ dx^-}{dy^+ dy^-},$$

since we know that the vacuum value of σ in Kruskal coordinates is zero. The remaining steps of determining the initial conditions follow from choosing the scalar functions. For the classical scalar matter we choose a “bump” function

$$f(y^+)|_{y^-=0} = \begin{cases} 0 & \text{if } x^+(y^+) < x_l^+, \\ \frac{H}{2} \cos^2(g(y^+)) e^{2ig(y^+) + \pi i} & \text{if } x_l^+ < x^+(y^+) < x_r^+, \\ 0 & \text{if } x_r^+ < x^+(y^+). \end{cases}$$

where

$$g(y^+) = \frac{\pi(2x^+(y^+) - x_l^+ - x_r^+)}{2(x_r^+ - x_l^+)}.$$

Notice we multiply the bump by a simple complex function, this determines the amount of charge we put in the system. However when we do the calculations we rather change the charge of the particles to affect the amount of charge sent in. When we work with the quantum corrected model, the Z field actually determines

the charge and so we must use a “soft step” function instead of the bump

$$Z(y^+)|_{y^-=0} = \begin{cases} 0 & \text{if } x^+(y^+) < x_l^+, \\ \frac{H}{2} \left(1 + \tanh \left(\tan(g(y^+)) \right) \right) & \text{if } x_l^+ < x^+(y^+) < x_r^+, \\ H & \text{if } x_r^+ < x^+(y^+). \end{cases}$$

In both cases we have three parameters, x_l^+, x_r^+ which determine where the matter enters and H which is the height of the bump or the step.

3.2 The Numerical Solver

Although theoretically the wave equations suffice to determine the fields inside Ω , given that the initial conditions satisfy the constraints, a numerical solver based on this method is hard to make stable. Instead we use the constraints along with the wave equations to integrate the system.

We discretize Ω into a square grid with step-length h in both directions. We label each grid-point by two numbers (i, j) so that any field f has the value $f_{(i,j)}$ at the point $y^+ = ih, y^- = jh$. By using the initial data described above we find the values for all fields at the grid-points $(i, 0)$ and $(0, j)$ where i, j run from 0 to $1/h$. Integration on the boundary is done by a fourth order Runge-Kutta scheme [18]. Then we run over the grid in a systematic fashion and use the algorithm below to solve the equations.

Following Hamade and Stewart [19] we write all the equations as systems of first order ODE’s in the y^+ and y^- directions so that we have two systems which depend on each other. Take for example the original CGHS model (2.5-2.6) where $S_M = F_{\mu\nu} = 0$

$$\partial_+ \begin{bmatrix} \psi \\ \sigma \\ \partial_+ \psi \\ \partial_- \psi \\ \partial_- \sigma \end{bmatrix} = \begin{bmatrix} \partial_+ \psi \\ \partial_+ \sigma \\ \partial_+ \sigma \partial_+ \psi \\ -\lambda^2 e^\sigma \\ 0 \end{bmatrix} \quad \partial_- \begin{bmatrix} \psi \\ \sigma \\ \partial_+ \psi \\ \partial_- \psi \\ \partial_+ \sigma \end{bmatrix} = \begin{bmatrix} \partial_- \psi \\ \partial_- \sigma \\ -\lambda^2 e^\sigma \\ \partial_- \sigma \partial_- \psi \\ 0 \end{bmatrix}$$

These systems are dependent on each other but together overdetermined. We write a general system as

$$\partial_+ u = f(u, v) \quad \partial_- v = g(u, v),$$

where u and v are vectors of fields and their derivatives. The two vectors may contain the same field, but when doing the calculation we make sure that elements representing the same field have the same value. We solve each of these systems using a predictor-corrector scheme [18] and while solving we average the calculated fields. Let's assume we want to calculate the values of u, v at the point (i, j) and that we know the values at all preceding points, that is (k, l) where $0 \leq k < i$ and $0 \leq l < j$. Then we can find an approximate solution to the system as follows,

1. Use a two step Adams-Bashforth method to calculate approximate value at (i, j)

$$\begin{aligned} v_{(i,j)}^* &= v_{(i-1,j)} + \frac{h}{2} (3f(u_{(i-1,j)}, v_{(i-1,j)}) - f(u_{(i-2,j)}, v_{(i-2,j)})), \\ u_{(i,j)}^* &= u_{(i,j-1)} + \frac{h}{2} (3g(u_{(i,j-1)}, v_{(i,j-1)}) - g(u_{(i,j-2)}, v_{(i,j-2)})). \end{aligned}$$

2. For those elements of the $u_{(i,j)}^*$ and $v_{(i,j)}^*$ vectors which correspond to the same field (for instance ψ) we replace in both vectors by the average of the two numbers. This should ensure a more stable calculation.

3. Two step Adams-Moulton Corrector is used to correct the preceding steps

$$\begin{aligned} v_{(i,j)} &= \frac{1}{6}v_{(i,j)}^* + 5 \left(v_{(i-1,j)} + \frac{h}{12} (f(u_{(i,j)}^*, v_{(i,j)}^*) + f(u_{(i-1,j)}, v_{(i-1,j)})) \right), \\ u_{(i,j)} &= \frac{1}{6}u_{(i,j)}^* + 5 \left(u_{(i-1,j)} + \frac{h}{12} (g(u_{(i,j)}^*, v_{(i,j)}^*) + g(u_{(i-1,j)}, v_{(i-1,j)})) \right). \end{aligned}$$

4. We average the numbers just as before.

While doing the integration we calculate auxiliary variables like the charge and the mass but we also locate the singularity at $\psi = 0$ and the apparent horizon at $\partial_+ \psi = 0$.

Testing of the numerics is performed by comparing numerical solutions to the known analytic solutions to the original CGHS model. That is the classical scalar field model without any charge. We find that for grid of size $10^3 \times 10^3$ we have relative error of order $10^{-4} - 10^{-3}$. We also take note of the convergence of the solutions to which there are no known analytic solutions. In general we find a grid of size $10^4 \times 10^4$ is good enough.

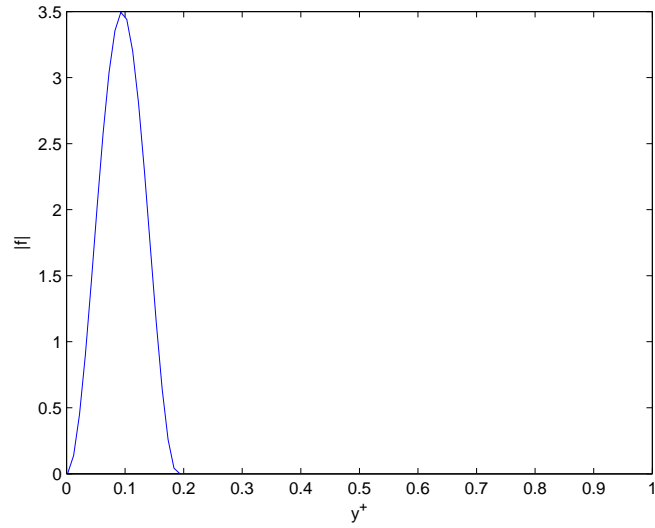


Figure 3.2: Shows the profile of the classical scalar field along the Σ^+ boundary.

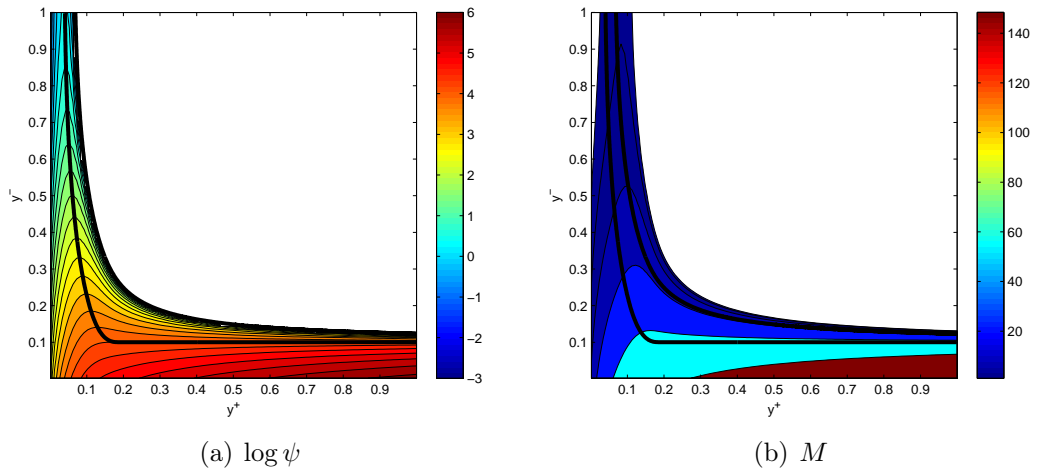


Figure 3.3: Contour plot of $\log \psi$ and the mass for a black hole formed by collapse of uncharged classical scalar matter. The upper thick line marks the singularity and the lower one, the horizon.

3.3 Results

We now present results obtained from numerical calculations done as described above.

In the following the mappings are chosen as $x^+ = 1 + 100y^+$ and $x^- = -0.3 - 10e^{-5y^-}$, this is to compress the uninteresting area below the singularity, and expand the interior of the black hole. We are not concerned with producing large black hole as of yet, and thus we choose only linear mapping in the x^+ direction. First we will establish the mass by letting the incoming classical scalar matter to be uncharged. The profile of the ingoing matter along the boundary Σ^+ can be seen on figure 3.2. For this specific profile we get a black hole of mass ~ 64 . A contour plot of the area function ψ and the mass function appear in figure 3.3. On all subsequent figures black thick lines will denote the apparent horizon located where $\partial_+\psi = 0$ and the singularity located at $\psi = 0$ of the corresponding black hole. The results are standard and easily comparable to the analytic solution as discussed above. Note that the mass decreases as we approach the singularity.

Next, we leave all parameters of the black hole unchanged except the charge of the scalar field, we choose $e = 1$. A contour plot of the area function can be found in figure 3.4. We instantly notice change from the uncharged case, here the singularity turns null inside the black hole and is "parallel" to the horizon. This opens up a gap at $x^+ \rightarrow \infty$ through which an observer could escape without having to visit the singularity. This is precisely analogous to the mass inflation scenario Poisson and Israel discovered inside charged black holes in 1+3 dimensions [6]. This shows that the CGHS model coupled to $U(1)$ as well as classical charged matter does possess similar properties as the full 1+3 dimensional theory. In fact when we take a look at the mass function we see that inside the black hole the mass increases exponentially (see figure 3.5).

When we include quantum effects in the calculation we switch to the quantum corrected equations (2.14-2.15) but we also have to modify our initial conditions, now the initial profile is not a "bump" but rather a "step". The initial profile can be seen on figure 3.6. The same basic procedure is taken when doing this calculation. We leave the charge as $e = 1$ but change the height of the step to match the mass of the above holes. The contour plot of the area function ψ (see figure 3.7) shows a different behaviour when compared to the classical case, the singularity does take a turn as in the classical case but is never null, the gap formed in the classical case starts to close up and disappears when we reach $x^+ \rightarrow \infty$. The reason is that the charge is evaporating as can be seen on figure 3.8. The mass function does still

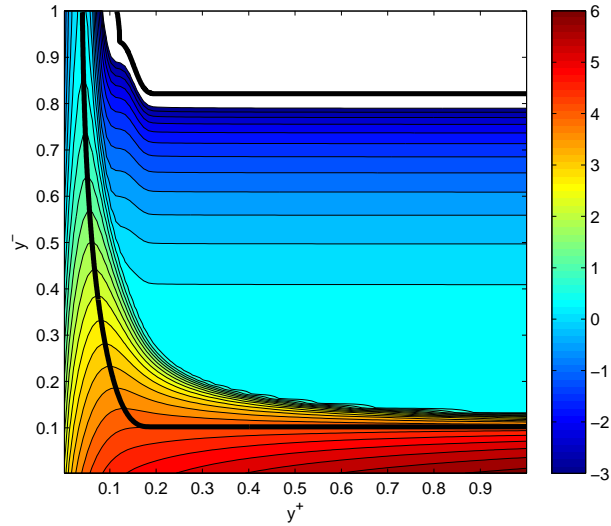


Figure 3.4: Contour plot of $\log \psi$ for a black hole formed by the collapse of a scalar field with charge $e = 1$. The kink in the contours close to the singularity is due to strong back reaction to the charge that has been forced in.

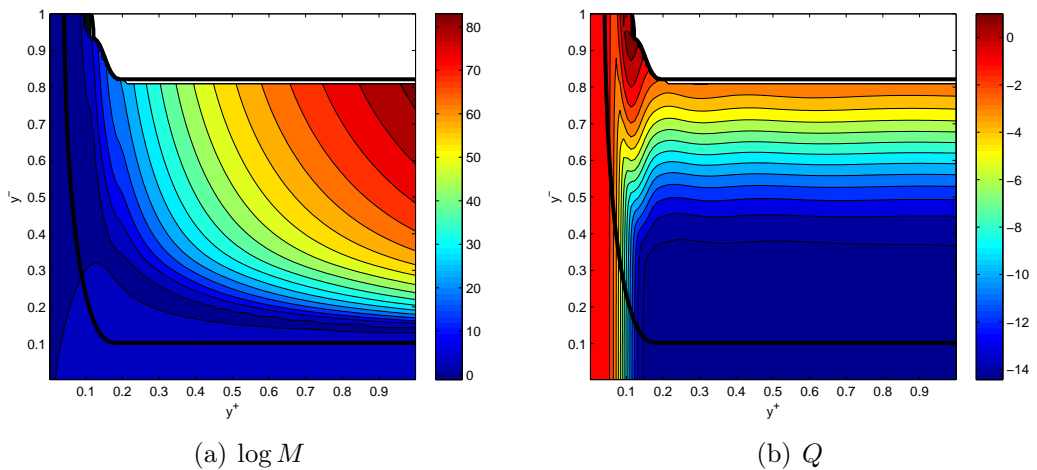


Figure 3.5: Contour plot of the logarithm of the mass function and the charge for the classical black hole formed by the collapse of charged scalar matter.

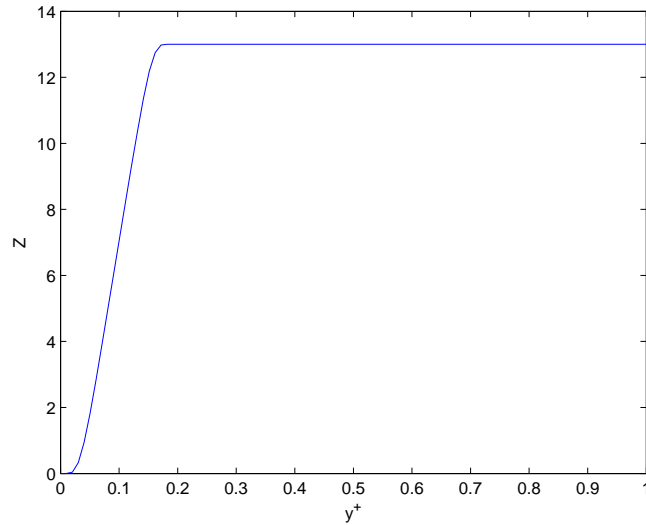


Figure 3.6: Shows the profile of the bosonized fermion field along the Σ^+ boundary.

increase as in the classical case (see figure 3.8).

It can be educational to look at the outgoing null rays plotted in the $\psi - y^+$ plane (see figure 3.9). We notice that the rays inside the black hole do not travel along the singularity as would be the signal of Cauchy horizon but rather they are attracted to the singularity.

For all above calculations we have used fields of zero mass. The qualitative behaviour of the solutions do not change if the mass of the fields is small compared to the charge.

3.4 Large Black Hole

Finally we push our method a little further by increasing the mass of the black hole. Now we will only show results for the quantum corrected model. The mappings employed this time are

$$x^+ = 1.5849e^{8y^+} \quad \text{and} \quad x^- = -68.6512 - 100e^{-30y^-}.$$

and are much more violent then before. Notice that the x^- mapping is constructed so that only the region inside and very close to the black hole will be explored. However when $y^- = 0$ we need ψ to be large compared to the matter fields and so we choose to multiply the exponent by -100 . The mass of the black hole is ~ 500 .

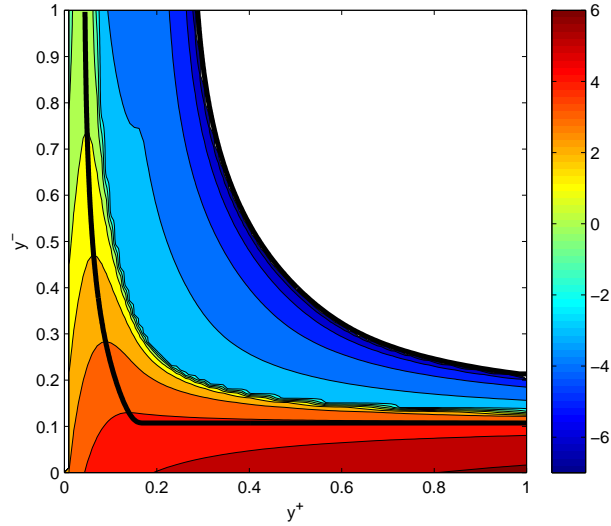


Figure 3.7: Contour plot of $\log \psi$ for a black hole formed by a collapse of bosonized fermion field with charge $e = 1$.

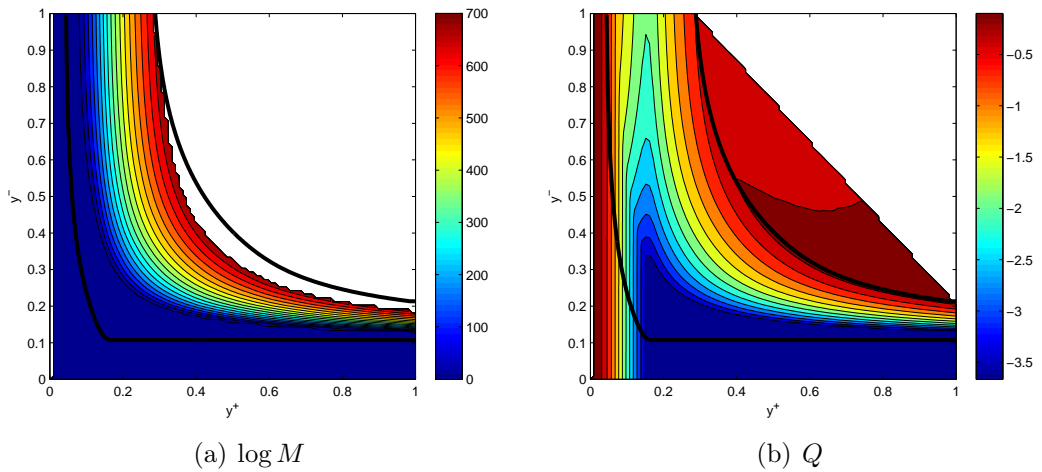


Figure 3.8: Contour plot of the logarithm of the mass function and the charge for a black hole formed by the collapse of charged bosonized fermions. The artifacts close to the singularity in the figures are due to the interpolation involved during plotting.

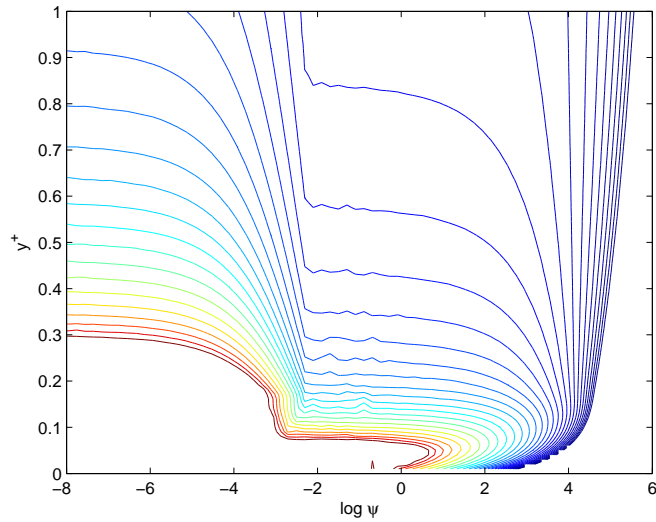


Figure 3.9: Outgoing null rays in the $\log \psi - y^+$ plane. The outermost escape to infinite area function, but the others approach the singularity at $\psi = 0$. This figure is for black hole of mass ~ 64 formed by charged bosonized fermions.

We choose the charge of the particles to be $e = 5$ and the mass to be zero. Very similar situation as for the smaller black holes can be observed in figures 3.10- 3.11. The charge evaporates quickly when the singularity is approached and the singularity converges to the horizon. The gap between the horizon and the singularity at the endpoint of calculation is of order unity in Kruskal coordinates. This shows that the same qualitative behaviour is present for large black holes as for the smaller ones.

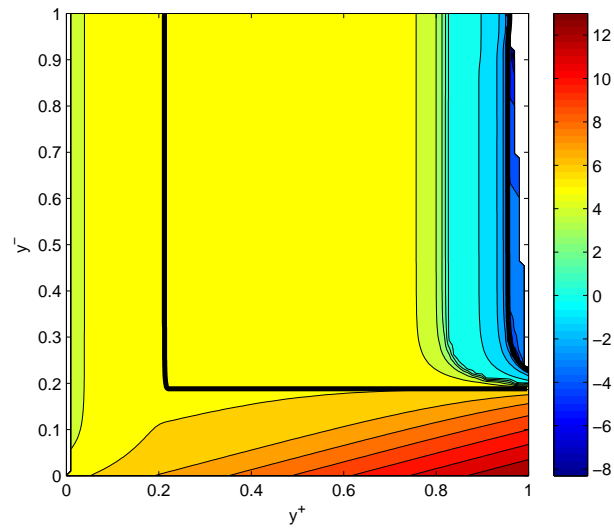


Figure 3.10: Shows a contour plot of $\log \psi$ for a large black hole formed by a collapse of bosonized fermion field with charge $e = 5$.

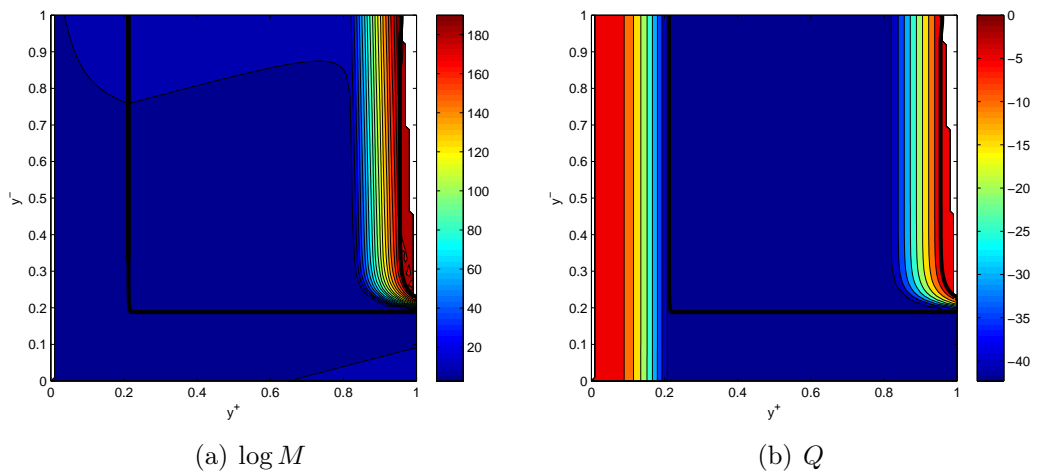


Figure 3.11: Contour plot of the logarithm of the mass function and the charge for a large black hole formed by the collapse of charged bosonized fermions.

4

Conclusions

By introducing a 1+1 dimensional dilaton gravity model coupled to an electric field we have been able to address some questions on the problem of the inner horizon of a charged black hole. We have seen signals of a Cauchy horizon in the classical case, confirming mass inflation inside a charged black hole. When quantum effects are introduced the charge of the black hole is no longer stable. The gap that opened up between the horizon and the singularity in the classical case closes so that an observer can not escape the singularity. This is an interesting result even though black holes in our universe do probably not carry charge [20]. They do however very likely carry angular momentum and the Kerr solution, that describes stationary spinning black holes [1], has many of the same properties as the Reissner-Nordström solution. It has been shown that the angular momentum of a spinning black hole evaporates when quantum effects are included [22]. Hence a similar change to the causal structure may appear as we have seen for charged black holes.

A more realistic scenario than the one considered in this thesis includes the evaporation of the mass of the black hole via the Hawking effect. Then the black hole has only finite lifetime. It then becomes a question of which quantum effect is stronger, that is if the black hole approaches extremal limit early in the process where the mass equals the charge of the black hole or if the black hole first evaporates the charge and continues uncharged.

The Hawking effect can be modelled in two dimensions [16] although numerics become quite hard technically. The strength of the Hawking effect is controlled by the number of flavors N in the model which must be large so that the approximations made are valid. For the same reason we need the mass of the black hole to be large compared to N [14, 15]. Now the evaporation time of the black hole is of order

e^M in Kruskal coordinates where M is the mass of the black hole [21]. Using the conformal mappings as we have done, it might be possible to reach the endpoint for macroscopic black holes. We would however have to use some methods to deal with very large values of ψ appearing in the lower right corner of Ω (see for example figure 3.10). It may be possible to use approximations since the equations are weakly coupled when ψ is large. It would be very interesting to continue in this direction since much of the work involving the numerics has been completed here.

Bibliography

- [1] R. d’Inverno, *Introducing Einstein’s Relativity*, Oxford University Press Inc., New York, 1992.
- [2] L. Thorlacius, “Cosmic censorship inside black holes,” *J. Korean Phys. Soc.* **50**, S1 (2007) [hep-th/0607048](#).
- [3] R. Penrose in *Battelle Recontres*, C. M. DeWitt and J. A. Wheeler, eds., pp. 121–235. W. A. Benjamin, New York, 1968.
- [4] R. A. Matzner, N. Zamorano and V. D. Sanberg, “Instability of the Cauchy horizon of Reissner–Nordström black holes,” *Phys. Rev.* **D19**(1979) 2821.
- [5] S. Chandrasekhar and J. Hartle, “On crossing the Cauchy horizon of a Reissner–Nordström black-hole,” *Proc. Roy. Soc. Lond.* **A384**(1982) 301.
- [6] E. Poisson and W. Israel, “Internal structure of black holes,” *Phys. Rev.* **D41** (1990) 1796–1809.
- [7] A. Ori, “Inner structure of a charged black hole: An exact mass-inflation solution,” *Phys. Rev. Lett* **67** (1991), no. 7, 789.
- [8] F. J. Tipler, "Singularity and causality violation", *Anal. Phys.* **108** (1977) 1–36.
- [9] S. Hod and T. Piran, “Mass-inflation in dynamical gravitational collapse of a charged scalar-field,” *Phys. Rev. Lett.* **81** (1998) 1554–1557, [gr-qc/9803004](#).
- [10] S. W. Hawking, “Particle creation by black holes,” *Commun. Math. Phys.* **43** (1975) 199–220.
- [11] J. S. Schwinger, “On gauge invariance and vacuum polarization,” *Phys. Rev.* **82** (1951) 664–679.

-
- [12] R. Herman and W. A. Hiscock, “Discharge of a Reissner–Nordström black hole within the event horizon,” *Phys. Rev.* **D49** (1994) 3946–3950.
- [13] E. Sorkin and T. Piran, “The effects of pair creation on charged gravitational collapse,” *Phys. Rev.* **D63** (2001) 084006, [gr-qc/0009095](#).
- [14] A. V. Frolov, K. R. Kristjansson and L. Thorlacius, “Semi-classical geometry of charged black holes,” *Phys. Rev.* **D72** (2005) 047, [hep-th/0504073](#).
- [15] A. V. Frolov, K. R. Kristjansson and L. Thorlacius, “Global geometry of two-dimensional charged black holes,” *Phys. Rev.* **D73** (2006) 124036, [hep-th/0604041](#).
- [16] C. G. Callan, S. B. Giddings, J. A. Harvey and A. Strominger, “Evanescent black holes,” *Phys. Rev.* **D45** (1992) 1005, [hep-th/9111056](#).
- [17] V. P. Frolov, “Two-dimensional black hole physics,” *Phys. Rev.* **D46** (1992) 5383.
- [18] B. Bradie, *A Friendly Introduction to Numerical Analysis*, Pearson Education, Inc., New Jersey, 2006.
- [19] R. S. Hamade and J. M. Stewart, “The spherically symmetric collapse of a massless scalar field,” *Class. Quant. Grav.* **13** (1996) 497, [gr-qc/9506044](#).
- [20] R. M. Wald, *General Relativity*, The University of Chicago Press, Chicago, USA, 1984.
- [21] J. G. Russo, L. Susskind and L. Thorlacius, “Black hole evaporation in 1+1 dimensions,” *Phys. Lett.* **B292** (1992) 13, [hep-th/9201074](#).
- [22] D. N. Page, “Particle Emission Rates From A Black Hole. 2. Massless Particles From A Rotating Hole,” *Phys. Rev. D* **14** (1976) 3260.