TOWARDS AN ELEMENTARY THEORY OF DRAINAGE BASIN EVOLUTION:

III. The Emergence of Random River Networks and Scaling Laws of Fluvial Landscapes

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ABSTRACT

The analysis of a family of physically-based landscape models supports a theory in which the evolution of fluvial landscapes dominated by advective sediment transport occurs in four distinct phases. In a short, first phase, there is an exponential increase away from unstable initial surfaces. The noise driving this process originates from the nonlinear amplification of small, random variations in both the surface and the rainfall. Nonlinear saturation of this process leads to a second phase during which the erosion process is driven by white noise generated by shock waves in the flow of water over convex portions of the eroding surfaces. The cumulative effect of the shocks is to drive a gradient sediment flow, leading to the emergence of an initial drainage pattern on the unstable surfaces. The process of channel emergence may be characterized in terms of a random walk. The erosional process of this phase is also characterized by the emergence of a fundamental scaling of the height-height correlation function of both the water depth h and the gradient of the water surface ∇H with an exponent value of 0.5. A third phase, occurring at a longer time scale, involves a self-similar cascade towards decaying separable solutions and an associated optimality principle. The erosion process in this phase is driven by noise generated by shocks originating in the flow of water over the stable, concave surfaces that characterize the separable solutions. These shocks are initiated by singularities of the surfaces at

the top of concave ridges and propagate downstream. The resulting noise, which is made spatially inhomogeneous by the nonlinear quenching in the sediment flow, drives the erosion process in a manner that differs from the process in the first two phases. This lead to a different scaling of the height-height correlation function for the water surface H and its gradient ∇H with an exponent value of 0.7. A fourth and final phase involves a short time scale over which the separable solutions collapse as the potential energy in the system approaches small values and the *global attractor* $H = 0$ exerts its influence. The values of the two scaling exponents, which we interpret as reflecting universal, but distinct, physical mechanisms involving diffusion driven by noise, correspond well with field measurements from areas for which the advective sediment transport processes of our models are applicable. Various other scaling laws, such as Hack's law and the Law of Exceedence Probabilities, are shown to result from the 0.7 scaling of the third stable phase, with exponent values (0.6 and $-0,4$, respectively) that are in good agreement with observation. Our results provide, as far as we know, the first derivation of such scaling results and laws from physically-based PDE models. The origin of the first scaling of 0.5 and the characterization of the initial channelization in term of random walks provide an important bridge between physically-based PDE models and discrete, stochastic models, particularly those based on random walks. While the latter are known to result in physically-realistic fluvial networks, their underlying assumptions have never before been justified in terms of physically-based PDE models.

1 Introduction

In two preceding papers [58, 61], we showed that a family of landscape evolution models capture, at least to a first approximation, the emergence and development of stable, dendritic patterns of valleys and ridges. In this paper we continue our investigation and demonstrate the manner in which these models also capture the effects of random influences in driving the processes of landscape evolution. In particular, our results provide a physical basis for explaining various fundamental scaling relationships that characterize fluvial landscapes and suggest a bridge between deterministic and stochastic theories of drainage basin evolution.

Two important lines in the development of theories of fluvial landscapes include the analysis of partial differential equations (PDEs) representing conservation conditions and the analysis of discrete stochastic models representing the manner in which random influences determine the configuration of drainage networks. The main contributions of the deterministic approach have been in explaining the fundamental instabilities that underlie channel forming processes (see, for example, [59, 15, 19, 58, 61, 3]) and the drive towards mature landsurfaces obeying optimality conditions and exhibiting self-similar evolution (see, for example, [12, 58, 61, 19, 20, 21, 18]). The main contributions of the stochastic approach have been in showing how theories based on simple statistical assumptions are capable of generating representations of drainage basins that are in accordance with observation (see Iturbe-Rodriguez and Rinaldo [28] for a general summary and Shreve [47, 48, 49], in particular, for an analysis of random topology models.)

An important criticism of the discrete stochastic models is that, despite their impressive simulations of many planar features of fluvial landscapes, they contain no acceptable bridge to the more physically acceptable theories based on PDEs. The complexity of fluvial processes and landforms, however, has long provided a justification for models that incorporate stochastic elements. The growing body of research focused on scaling relationships in landscapes, recently summarized in [28], emphasizes the importance of integrating deterministic and stochastic representations of fluvial processes into a tractable and useful theory. A general goal of this paper, therefore, is to show that by interpreting PDE-based models of landscape evolution as driven by random influences, one obtains significant insights not only into the nature of landscape scaling laws, but also into the manner in which the deterministic and stochastic lines of investigation may be bridged.

The contribution of the current paper towards this general goal is based on an analysis of how the models investigated in [58, 61] respond when driven by random influences. In particular, we show how such models provide insight into two important and open problems of geomorphic research, namely the nature of the physical processes that underly the emergence of branching networks of channels from relatively unstructured surfaces and the

emergence of the fundamental scaling relations that characterize fluvial landscapes.

An important and open question of geomorphic research concerns the emergence and stabilization of organized patterns of fluvial erosion. As far as we are aware, the results presented in [58, 61] provide the only demonstration so far that solutions to PDEs representing physically-reasonable conservation conditions model the emergence of stable, dendritic patterns of fluvial erosion from unstructured surfaces *without* the need to introduce non-physical constructs. While Willgoose et. al. [20, 21], for example, employed PDEs representing conservation conditions to model the emergence of channelized forms, their analysis required the artificial seeding of the locations at which channels formed (for a critique, see [65]) in order to produce solutions interpretable as dendritic networks. Other researchers investigating similar models (see, for example, [59, 15, 3]) have only shown that such models possess the necessary channel-growing instabilities in the linear regime. Although various researchers have employed similar models in explaining landscape response to environmental conditions (see, for example, [45, 24, 23, 22]), their analysis has not focused on explaining the fundamental mechanisms of channel emergence and stabilization.

A second open question relates to the construction of physically-based explanations for the many interrelated scaling laws that characterize fluvial landscapes. As described in a large and growing literature summarized in [28], such laws have been found to hold at the level of the river channel (e.g., *the laws of hydraulic geometry*); at the level of the drainage basin (e.g., *Horton's laws* and *Hack's law*); and at the level of whole landscapes (e.g., the scaling laws associated with various *structure functions*). The fundamental issue raised by such scaling relationships is their relationship to the dynamics and physical processes underlying drainage basin evolution. While there are no results of which we are aware that provide physical explanations for the emergent scalings, there is growing evidence that many of these relationships are explainable in terms of: (1) formative physical processes, such as water and sediment transport; (2) universal features in the underlying mechanisms responsible for the evolution of spatial structures; and (3) inherent randomness associated with the instabilities of nonlinear dynamical processes (see, for example, [57, 1, 63, 13, 14, 6, 50, 2, 7]). Hence we hypothesize that *most, if not all, of the fundamental scalings of fluvial landscapes can be understood on the basis of the underlying physical processes and/or universal features of the underlying nonlinear dynamics.*

We address these two issues within the modeling context presented in [58, 61] and within an analytical context that is provided by viewing fluvial landscapes as complex surfaces whose evolution is determined by nonlinear processes driven by random inputs. In relation to our modeling context, we note that the models described in Section 2 below focus on the *advective entrainment and transport of sediment in transport limited conditions* as the dominant process in drainage basin evolution. While other processes, such as detachment

limited entrainment and gravity driven diffusion, are important processes in many geomorphic contexts, the modeling results presented in [58, 61] indicate that advective entrainment and transport processes are in some sense the fundamental processes underlying drainage basin evolution. We believe that isolating and understanding the effects of these processes is a necessary and critical step in understanding the effects of many interacting processes on landscape development.

In relation to our analytical context, we apply the methods developed in a large and rapidly growing body of research focused on explaining the dynamics and forms of complex surfaces and interfaces evolving in a variety of physical contexts. Important goals of this literature are to characterize the qualitative behavior of high- and infinite-dimensional, nonlinear systems that are driven by noise and to explain the origin of temporal and spatial scaling behavior in a wide variety of phenomena [39, 25, 31, 9]. It is now known that the evolution of surfaces whose dynamics are driven by various forms of noise are frequently characterized by *scaling laws*. Such laws often, but not always, indicate that the system is insensitive to differences in the details of the underlying mechanisms and processes. This insensitivity leads to the observation that seemingly different systems exhibit qualitatively similar, or *universality classes* of, behaviors (see, for example, Sneppen [33].) This fact of great value insofar as it allows one to extend results that are known about the response of one system to noisy inputs to other systems in the same universality class. In the following analysis, we make use of such extensions in understanding how the two types of noise (homogeneous and quenched) lead to distinct emergent scalings in the eroding surfaces described by our models.

We employ several specific techniques from the emerging theory of complex surface evolution in investigating the models discussed in [58, 61] as systems driven by noise or stochastic processes. First, we characterize the statistical structure of eroding surfaces and flows in terms of various *structure functions* that represent the statistical correlation structure of complex surfaces. Second, we apply known results from this theory concerning the form of scalings that emerge from appropriate universality classes of PDEs when subjected to random driving forces of a specific form. As noted above, the rationale for such application is that systems belonging to the same universality class manifest qualitatively similar behaviors. In order to take advantage of these known results we employ suitable approximations of our models in conjunction with specific models of noise that we justify in terms of the basic properties of our models. We also connect part of our theory to the concept of self-organized criticality (SOC) as proposed by Bak et al. [44, 43, 38, 55]. Although this concept has yet to find an appropriate mathematical formulation, it is based on the underlying idea of an *attractor* poised at criticality to which a system is driven by random perturbations. Such systems are characterized by essential relationships between the spatial and the temporal scalings of events [62] and by specific asymptotic forms of the structure

functions [1].

In relation to the statistical structure of our eroding surfaces and flows, our main results are the discovery of two fundamental scaling relationships that are represented in terms of the asymptotic forms of a structure function characterizing the *width*, or height-height correlation function (see section 2.3 below), of eroding surfaces. A first scaling relation, with an exponent of 0.5 characterizing the depth of water flow and the slope of the water surface, emerges early. This scaling is fundamentally associated with the development of channelized flows on unstable surfaces and indicates the importance of stochastic influences in determining the configuration of emerging channel networks. A second scaling, with an exponent of approximately 0.7, is associated with self-similar separable solutions which are associated with the *mature* stages of development and the emergence of a variational principle. This scaling characterizes the form of the water and land surfaces. The differences between the two scalings suggest that they are determined by different mechanisms.

We note with great interest that field values of the same height-height correlation function obtained for landscapes in Ethiopia, Somalia, and Saudi Arabia by Weissel and Pratson [32], fall neatly into the range 0.5-0.7. Although empirical scaling results of this form tend to cluster around a relatively small set of values, these results provide some validation for our theory, particularly since our models may be interpreted as representing conditions under which our advective models of sediment entrainment and transport apply, since the diffusive mass-wasting processes of humid landscapes are relatively unimportant. We may therefore interpret our models as providing physically-based explanations for such empirical scaling relationships.

In relation to the application of known results from the theory of complex surfaces to explain the physical mechanisms underlying these scalings, we first construct appropriate linearizations of our models in the two relevant regimes: the early channel initiation regime and the mature, separable surface regime. We show that, when viewed at their fast time scales, these linearized models possess noise generating mechanisms. These mechanisms act as sources of noise that drive the erosion equations at their long time scales and thereby produce the observed scalings. We may interpret the initial, small, random variations as characteristic of the eroding elevations and of the rainfall process [2].

The mechanisms that generate this driving noise are important since they provide a linkage between the observed scalings and the physical mechanisms underlying the evolution of the eroding surfaces. We identify two plausible mechanisms of noise generation that stem from instabilities in the flow of water over the eroding surfaces and that take place on

[1] We believe, in fact, that our theory may provide some of the foundation for a *mathematical theory* of self-organized criticality.

[2] In computing solutions to our basic model, we simulate such noise as a combination of perturbations to the initial eroding surface and the round-off error in the numerical computations.

the short time-scales associated with such flows. A mechanism that we associate with the regime of *early channel initiation* involves the formation of *shocks* [3] in the flow of water over locally convex portions of landscapes in which channels are first emerging. These shock formations are of two types. The first is associated with nonlinearities in the water flow, the second with both nonlinearities in the sediment flow and the linear propagation of singularities in the water flow. As we show below, noise generated by shocks of both types may be viewed as driving the process of channelization. A second mechanism of noise generation that we associate with the regime of *mature, separable surfaces* again takes place on a short time-scale and involves the formation of shocks on characteristically concave surfaces at the points where the derivatives of the surface elevation function are singular. While we emphasize that these mechanisms of nonlinear noise generation are specific to our simple advective models, mechanisms involving analogous effects almost certainly exist in more complex models.

The cumulative effects of the noise generated in the flow of water at short time-scales impact the sediment flows at the longer time-scale of erosion and lead to the observed scalings. Hence we view the first scaling relation, which arises from an erosion process driven by noise generated by shocks in the flow of water, as resulting from universal phenomena that relate to diffusive (random) processes driven by homogeneous noise. Since this scaling is associated with the development of channelized flows on unstable surfaces, it indicates the importance of stochastic influences in determining the configuration of emerging channel networks. In particular, we show that such configurations may be viewed as arising from a random walk. The second scaling relation, which is arises from erosion processes being driven by noise generated by shocks in the flow of water over separable surfaces with singularities [58], is also a universal phenomenon arising in a noise-driven diffusive process. Whereas the noise generated by shocks over the initial channelizing surfaces is homogeneous, the noise generated by the shock-inducing singularities of the separable surfaces is highly inhomogeneous due to nonlinear quenching. Hence, although the two processes are similar in both being diffusive and noise-driven, the differences in forms of the associated noise result in differing scaling exponents.

We show that various other scaling relations follow from our two basic scaling relations. In particular, we show that the second scaling leads to a simple derivation of the established scaling relationship between stream length and drainage area known as *Hack's Law*. Furthermore, the noise that leads to the first scaling also drives the initial growth of channels, allowing us to characterize, in terms of random walks, the nature of the process by which channels emerge from unstructured surfaces. We emphasize that our contributions in relation to our scaling results are not simply in showing how one particular model gives rise to

[3] A shock is a discontinuity in a wave-like form that occurs when the higher parts of the wave propagate at a higher velocity than the lower part (see, for example, Smoller [54]).

(self-similar) scaling, but to show that the scaling of surfaces and processes in a large class of relevant models may be viewed as being caused by universal physical processes.

Based on these results, the findings of our two previous papers [58, 61], and previous research on the initial instabilities of such models, we suggest a theory of the evolution of landscapes described by our advective models. In particular, this theory characterizes landscape development in terms of an evolutionary sequence involving four stages and associated time scales. The first stage occurs over a relatively short time scale and is characterized by an initial, exponential increase away from the unstable initial surface (see [58]) and involves saturation by nonlinearities. In a second stage, occurring over an intermediate time-scale, the mechanisms underlying the first scaling of 0.5 become effective and a channelized network that may be interpreted in terms of random walks emerges. A third stage, occurring at a longer time scale, involves a self-similar cascade towards decaying separable solutions driven by the mechanisms underlying the second scaling. Various scaling laws, such as Hack's law, also emerge during this stage. Finally, a fourth and last stage involves a short time scale over which the separable solutions collapse as the potential energy in the system approaches small values and the "global attractor" $H = 0$ exerts its influence. We note with interest that, in terms of the Davisian model of erosion cycles [66], the first two stages may be interpreted as corresponding to *youth*, the third stage to *maturity*, and the final stage to *old age.*

Many details of this theory remain for investigation and significant efforts will be required to generalize the theory to cases involving other geomorphic processes. Specific questions of interest include whether the two suggested mechanisms of noise generation can be generalized to other, more comprehensive models; whether the scaling results can be shown to follow in the full nonlinear regime of the equations; and whether the two scalings are sufficient to determine all of the other scaling relations of fluvial landscapes, or whether there are other scalings that we have either yet to find or that arise as a result of processes not encompassed within our advective models. Despite these daunting research problems, we believe that the results presented in this paper provide a reasonable qualitative model of the emergence of form and scalings over a broad range of fluvial contexts and that such results are likely to be independent of many specific modeling details.

The paper is structured as follows. In Section 2 we discuss the mathematical models, the nonlinear shock formation that generates the noise, and the structure functions that give rise to the scaling relations. The results of numerical simulations of the mathematical models, which are presented in Section 3, provide a basis for the empirical and analytical investigations presented in the remaining sections of the paper. In Section 4, we provide a theoretical explanation for the first scaling relation that is associated with the emergence of channelization on unstructured surfaces. In Section 5, we provide a theoretical explanation for the second scaling relation that is associated with the separable landscapes. We discuss

the implications of our results for the understanding of geomorphic phenomena in Section 6, while Section 7 contains a summary of our results. A derivation of the relationship between Horton's laws, a proof of the formation of shocks from surface singularities and their linear propagation, and an explanation for the observed scaling of the gradient of the water surface are contained in Appendices D, B and C.

2 Modeling Eroding Surfaces and their Scaling Laws

Because the surfaces generated by fluvial erosion are the result of highly nonlinear processes driven by noisy inputs, they are complex and difficult to represent in traditional terms. We now discuss the models of nonlinear erosion processes from which our results are generated, and briefly indicate the nature and origin of the random effects that drive the equations. We then describe the manner in which such surfaces may be represented in terms of structure functions, the asymptotic forms of such structure functions, and the associated scaling laws.

2.1 A Family of Landscape Evolution Models

The scaling results presented in this paper are derived from the same family of models studied analytically and numerically by Smith, Birnir, and Merchant [58], Smith, Merchant, and Birnir [58], and Birnir, Smith, and Merchant [7]. These models represent the *advective entrainment and transport of sediment in transport limited* conditions [4], Based on our previous analyses and on the results reported in this paper, we believe that they capture essential aspects of fluvial erosion at small to medium scales of spatial resolution (see, for example, Howard, Dietrich, and Seidl [5]) [4]. While it is straightforward to extended these models to represent other processes that are significant in the evolution of fluvial landscapes, such as gravity-driven diffusion processes on hillslopes, we believe that their effects would mask the scaling relations associated with the advective processes of sediment transport in channel and overland flows [5].

We now provide a summary description of the family of models. A more detailed description of their derivation is provided in [58]. We also provide a brief discussion of those aspects of the characteristic time scales of the models that are relevant to the present context. The models are based on conditions describing the conservation of water and sediment fluxes over a continuous, erodible surface $z = z(x, y, t)$

$$-R + \frac{\partial h}{\partial t} = \nabla \cdot (\mathbf{u}_w \, q_w) \,, \tag{1}$$

[4] Whether they capture essential aspects at large scales of spatial resolution is an open issue.

[5] We plan to explore the effects of such processes on scaling in later papers.

$$\frac{\partial z}{\partial t} = \nabla \cdot (\mathbf{u}_w \, q_s) \tag{2}$$

in which $h = h(x, y, t)$ is the depth of water varying continuously over the landsurface, $-\mathbf{u}_w = -\frac{\nabla H}{|\nabla H|}$ is a unit vector in the direction of both water and the advected sediment flows, $H = H(x, y, t) = z(x, y, t) + h(x, y, t)$ is a free water surface, q_w represents the flux of water per unit width, and q_s represents the advected flux of sediment per unit width.

There are three natural time scales that characterize the dynamics of equations (1, (2). These time scales may be derived by transforming the variables of equations (1), (2) to dimensionless form, using relations $v = [v]v^*$ in which $[v]$ is a scale and v^* a dimensionless variable. Given values for the scaling parameters $[H], [h], [q_w], [q_s], [R]$ and $[x] = [y]$ that are characteristic of the variables in equations (1), (2), [6] we may then define representative values of the scaling parameter for the time variable ($[t]$) by setting the values of specific dimensionless parameters of the problem to unity.

A *short time scale* $[t]$ for t may be defined by the relation $[t][q_w]/[h][x] = 1$, which may be interpreted to imply that $[t]$ is the time scale at which the volume of water flowing over a lateral cross section of the ridge is approximately the same as the volume of water on the surface. Applying this scaling to dimensionless versions of equations (1), (2) and dropping the asterisks denoting dimensionless variables, we obtain

$$\begin{aligned} \frac{\partial h}{\partial t} &= \nabla \cdot (\mathbf{u}_w \, q_w) + \frac{[x][R]}{[q_w]} R \\ \frac{\partial H}{\partial t} - \frac{[h]}{[H]} \frac{\partial h}{\partial t} &= \frac{[h][q_s]}{[H][q_w]} \nabla \cdot (\mathbf{u}_w \, q_s) \, . \end{aligned}$$

It is natural to choose a scaling that makes the dimensionless parameter $[x][R]/[q_w] = 1$ in the water flow equation and interpret all terms in this equation as being $O(1)$ (see, for example, [3]). The significance of this fact, as we show in Section 2.4 and in Appendix A, is that the flows described by this equation are characterized by shock waves that we may interpret as short term noise. The order of the dimensionless parameter $[h][q_s]/[H][q_w]$ in the erosion equation is much smaller than unity and smaller than $[h]/[H]$. Hence we obtain the (approximate) erosion equation

$$\frac{\partial H}{\partial t} \approx \frac{[h]}{[H]} \frac{\partial h}{\partial t}$$

which indicates that variations in the water surface H are *small* at this time scale and driven by short term variations in the flow depth h described by the water flow equation. This equation also implies that erosion of the surface is negligible at this time scale.

[6] A useful set of such values characterize the representative scales when the basic model (1, 2) is used to represent the erosion of a small, linear ridge.

A time scale $[t]$ that is *intermediate* between the previous short scale and longer scales may be defined by the relation $[t][q_s]/[h][x] = 1$, which implies that we may interpret $[t]$ as a time scale at which the volume of sediment flowing over a lateral cross section of the ridge is approximately equal to the volume of water on the ridge at any time. On applying this scaling to equations (1), (2) we obtain

$$\begin{aligned} \frac{[q_s]}{[q_w]}\frac{\partial h}{\partial t} &= \nabla \cdot (\mathbf{u}_w\, q_w) + R \\ \frac{\partial H}{\partial t} &= \frac{[h]}{[H]}\nabla \cdot (\mathbf{u}_w\, q_s) + \frac{[h]}{[H]}\frac{\partial h}{\partial t} \end{aligned}$$

where we have once more used to the scaling relation $[x][R]/[q_w] = 1$. We note that the dimensionless parameter $[q_s]/[q_w]$ that multiplies the term $\frac{\partial h}{\partial t}$ has a magnitude that is greater than its magnitude at the short time scale. Since the evolution of the water surface H is determined by two terms that we may assume to be $O(1)$, see [3], and since both are multiplied by the same small dimensionless parameter $[h]/[H]$, we may interpret this to mean that changes in the surface H are determined to a significant degree by small fluctuations in the depth of water h, indicated by the water flow equation, as well as by the erosion of the z surface.

A *long time scale* $[t]$ may be defined by the relation $[t][q_s]/[H][x] = 1$ and we may interpret $[t]$ as the time scale at which the volume of sediment flowing over a lateral cross section of the ridge represents a significant proportion of the volume of the ridge. On applying this scaling to equations (1), (2) and again using the scaling relation $[x][R]/[q_w] = 1$, we obtain

$$\begin{aligned} \frac{[h][q_s]}{[H][q_w]}\frac{\partial h}{\partial t} &= \nabla \cdot (\mathbf{u}_w\, q_w) + \frac{[x][R]}{[q_w]}R \\ \frac{\partial H}{\partial t} &= \nabla \cdot (\mathbf{u}_w\, q_s) + \frac{[h]}{[H]}\frac{\partial h}{\partial t}. \end{aligned}$$

The small size of the dimensionless parameter on the LHS of the water flow equation, $[h][q_s]/[H][q_w] << 1$, suggests that the basic flow is essentially an equilibrium flow down the surface gradient. It is not unreasonable, therefore, to adopt the following approximation to the water flow relation

$$0 \approx \nabla \cdot (\mathbf{u}_w\, q_w) + \frac{[x][R]}{[q_w]}R \tag{3}$$

as was done, for example, in the analyses presented in [59, 15, 58, 61]. While we may interpret the dominant balance in the erosion equation as being between the first two terms, we may interpret the relatively small term $[h]/[H]\partial h/\partial t$ as representing random variations that drive the sediment flow.

2.2 Previous Results for Long Time Scale Approximations

Smith, Birnir, and Merchant [58, 61] analyzed a subfamily of the models (1), (2) that was obtained with the use of the following assumption. First, the long time scale approximation 3 to the basic model was assumed, together with the additional scaling relationship $[h]/[H] = [q_s]/[q_w] \equiv \eta$. Second, a Manning-type constitutive relation

$$q_w = nh^{\frac{5}{3}} \mid \nabla H \mid^{\frac{1}{2}} \tag{4}$$

was adopted to represent the flux of water. Third, a power law relation

$$q_s = kh^{\frac{5}{3}\alpha} \mid \nabla H \mid^{\frac{\beta}{2}+\delta} \tag{5}$$

was employed in representing the flux of sediment q_s [67]. These assumptions lead to the following dimensionless equations:

$$-R \; = \nabla \cdot \left[\frac{\nabla H}{\mid \nabla H \mid} h^{\frac{5}{3}} \mid \nabla H \mid^{\frac{1}{2}} \right] \tag{6}$$

$$\frac{\partial H}{\partial t} - \eta \frac{\partial h}{\partial t} \; = \nabla \cdot \left[\frac{\nabla H}{\mid \nabla H \mid} h^{\frac{5}{3}\gamma} \mid \nabla H \mid^{\frac{\gamma}{2}+\delta} \right]. \tag{7}$$

The initial and boundary conditions employed in these analyses modeled a small, linear ridge undergoing erosion as a result of a uniform rainfall (see equations (9), (10) below.)

Applying the numerical methods described in [61] to equations (6)-(7), it was shown [61] that initially unstructured but randomly perturbed two-dimensional surfaces are unstable when eroded by water. Channelized flows develop on the surfaces with a region of maximum channelization first emerging towards the lower boundary of surfaces with an initially planar configuration [7]. This process of channelization involves the merging of "rivulets" and the formation of larger "channels". The longitudinal profiles of the ridge simultaneously develop a concavity that also emerges initially near the base of the ridge under the influence of the fixed lower boundary. The region characterized by the maximum rate of channelization and by the emergence of the concavity gradually moves towards the upper boundary of the ridge as the surface erodes.

A pattern of dendritic valleys separated by ridges emerges and comes to dominate the eroding surface. Such a pattern is illustrated in Figures 1 and 2, which represent the surface

[7] A variety of numerical experiments have shown that channelization occurs on surfaces with a large array of initial configurations.

after 10% of an initially planar surface has been eroded. After a characteristic period of erosion, the surfaces evolve towards stable landscapes of concave valleys and ridges that develop in a self-similar manner. Such a pattern is illustrated in Figures 3 and 4, which represent the surface after 60% of an initial planar surface has been eroded [8].

The convergence to these forms occurs from initial surfaces that are both planar and non-planar. The characteristic period of erosion that precedes this convergence, which we measure in terms of the percentage of the original surface eroded, depends on the nature of the initial surface, the rainfall rate, and the parameters as γ and δ. These mature landscapes, consisting of stable patterns of concavities, valleys, ridges, and associated flows, are well-described by a class of solutions to the nonlinear PDEs (6)-(7) for cases in which $\eta \frac{\partial h}{\partial t}$ may be ignored in equation (7) relative to $\frac{\partial H}{\partial t}$ [58]. Such solutions depend on the characteristics of the water and sediment transport laws and on the boundary conditions for the PDEs; they are separable in time and space

$$H(x,y,t) \;=\; T_H(t) H_o(x,y), \qquad h(x,y,t) \;=\; T_h(t) h_o(x,y); \tag{8}$$

and they are stable [58]. They are also characterized in terms of a variational principle [58], by which a simple function of the sediment transport over the surface is minimized, subject to constraints involving the conservation of water flow and the elevations of the initial surface.

2.3 The Models Underlying the Present Analysis

The analysis presented in the current paper is similarly based on the use of equations (1), (2), the Manning-type constitutive relation 4 for the flux of water, a power law relation (5) for the flux of sediment, and the initial and boundary conditions prescribed in equations (11), (12). For reasons that we discuss below, however, *we no longer assume that the term $\frac{\partial h}{\partial t}$ in equation (1) is always negligible.* As we show this term leads to significant disturbances in the flow of water at the short time scales discussed in section 2.1. We interpret these disturbances as noise, whose cumulative effects, as we discuss below, are significant in driving the flow of sediment at the longer time scales.

We therefore focus our attention on the model

$$\eta^2 \frac{\partial h}{\partial t} \;=\; \nabla \cdot \left[\frac{\nabla H}{|\nabla H|} h^{\frac{5}{3}} |\nabla H|^{\frac{1}{2}} \right] \;+\; R, \tag{9}$$

[8] We note, in particular, the sharpness of the longitudinal ridges, or divides, separating the valleys.

$$\frac{\partial H}{\partial t} \; - \; \eta \frac{\partial h}{\partial t} \; = \nabla \cdot \left[\frac{\nabla H}{\mid \nabla H \mid} h^{\frac{5}{3}\gamma} \mid \nabla H \mid^{\frac{\gamma}{2}+\delta} \right] \tag{10}$$

in which we have adopted the scaling relation $[h]/[H] = [q_w]/[q_s] \equiv \eta$. In relation to the three time scales discussed above, we note that there is no loss of generality in this representation. In particular, we may obtain the short time scale version of the equations by applying the transformation $\tau = t/\eta^2$ and the intermediate time scale version by applying the the transformation $\tau = t/\eta$.

We employ the same boundary and initial conditions as [58, 61] to model a small linear ridge extending uniformly in the lateral (x)-direction and defined over a rectangular domain

$$D = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq W, \; 0 \leq y \leq L\},$$

with initial conditions

$$\begin{aligned} h(x, y, 0) &= d(y), \;\; d(L) = 0, \\ H(x, y, 0) &= cy, \quad 0 \leq y \leq L \end{aligned} \tag{11}$$

and boundary conditions

$$\begin{aligned} h(x, L, t) &= 0, \\ H(x, 0, t) = \; & h_0 \; = h(x, 0, t) \end{aligned} \tag{12}$$

corresponding to a water depth of zero at the top of the ridge and an absorbing body of water at the base of the ridge. While the water surface must be considered to be a free surface at the top of the ridge, it may be viewed as consisting of finitely many smooth curves that are solutions of a nonlinear ODE (the PDE restricted to the boundary). These curves are joined in a continuous, but not smooth, moving boundary (see, for example, Figure 3.) The upper boundary is characterized by the additional conditions

$$q_w = q_s = 0,$$

indicating the absence of any flux of water or sediment over this boundary. Since q_w and q_s are expressed as powers of h and ∇H in the constitutive relations, these conditions imply that the vanishing of the water depth dominates the blow-up of the gradient of H (in q_w and q_s) and that the normal derivative of H may become infinite at the upper boundary

$$n \cdot \nabla H(x, w, t) = \infty.$$

The boundary conditions on the lateral boundaries of the ridge at $x = 0$ and $x = W$ are taken to be periodic, modeling a linear ridge of infinite extent.

2.4 The Role of Noise and Its Generation by Shocks

The reasons for retaining the term $\eta^2 \frac{\partial h}{\partial t}$ in the water flow equation (9) is related to the issue of noise, its effect in driving the erosion process, and its relation to the scaling relations that characterize eroding land surfaces. Many investigations of PDEs that model the evolution of complex surfaces and interfaces have shown the significant role played by noisy inputs in determining the behavior of such surfaces. As shown, for example, by Sneppen [33] the nature of the noise driving the evolution of complex surfaces and interfaces is often a major determinant of the structure, and hence the scaling properties, in both emergent and mature systems.

It is especially essential to consider the effects of noise when modeling the evolution of geomorphic phenomena. The complexities of geomorphic processes and surfaces make it virtually impossible to model many aspects of landscape evolution without representing various classes of model inputs in stochastic terms. The nonlinear properties of such systems may lead to invalid analyses if explicit account is not taken of the stochastic nature of various inputs. This is especially the case for landscape phenomena whose evolution is governed by complex, interdependent processes characterized by differing time scales. Within the context of our problem, for example, it is difficult to imagine representations of either small scale (and hence short term) perturbations to the landsurface $z(x, y, t)$ or short term variations in rainfall R that are not stochastic. Given such inputs, the flow of water over the surface $z(x, y, t)$, as modeled by equation (9), is itself assured to be stochastic, as are any short term effects of such variations on the entrainment and transport of sediment. As we show below, however, random effects such as these that occur over relatively short time scales can neither be ignored nor averaged out at longer time scales, since their effects may accumulate in non-trivial ways.

Plausible sources for random inputs to the models presented in [58, 61] and represented in (9), (10) include, at the least, (1) the form of the initial surface; (2) the structure of the eroding substrate; and (3) the inputs of rainfall. In the investigations described in [61], and again in the present analysis, the initial surfaces from which channelized forms evolve are randomly perturbed. Not only is this necessary for breaking the symmetry of the surfaces, and hence allowing channel-forming instabilities to become effective, but accords with the preceding argument on the impossibility of modeling, in a deterministic manner, the processes and forms leading to small surface perturbations.

In the investigations described in [61], and again in the present analysis, neither the eroding substrate nor the rainfall inputs are represented in terms of stochastic effects. It is, therefore, valid to ask whether this apparent lack of time-dependent stochastic variation in the model inputs leads to any significant lack of generality in our modeling results. We do not believe this to be the case. First we note compelling empirical evidence from a variety of

studies (see, for example, Morisawa [42] and Schumm, Mosley, and Weaver [52]) indicating that the configuration of emergent drainage patterns is strongly determined by initial, and essentially random, surface configurations.

Second, and perhaps of greater intrinsic interest, is the fact that the mechanisms of water flow embedded in (9) are capable of generating variations in flows at short time scales that may be interpreted as constituting noise. In particular, we note that equation (9) is capable of generating *shock waves* that provide a continuous source of variation to the system. It is, for example, well-known that flows described by the analogous kinematic wave approximation of the Saint-Venant equations are characterized by shocks [36, 53]. The kinematic wave approximation is known to hold under conditions of overland flow on steep slopes and for variations in discharge that are of small amplitude or long wavelength (see, for example, [36, 53]).

In Appendix A we show that the water flow equation (9) becomes a one-dimensional equation describing the flow of water down the gradient of the water surface

$$\frac{\partial h}{\partial t} \; - \; \frac{5}{3\eta^2} h^{2/3} \mid \nabla H \mid^{\frac{1}{2}} \frac{\partial h}{\partial s} \; = \; \left[\frac{\Delta H - 1}{\mid \nabla H \mid^{\frac{1}{2}}} \right] \frac{h^{5/3}}{\eta^2} \; + \; \frac{R}{\eta^2}. \tag{13}$$

in which s parametrizes this direction. We provide a proof of the existence of shocks in the flows described this equation in Appendix A, and show below that such shocks lead to white noise that drives the flow of sediment (10). Two distinct mechanisms lead to such noise. First, on convex surfaces the water flow equation (13) has solutions that develop shocks. Second, the sediment flow equation (10) develops solutions that are not smooth and that possess singularities in a derivative of H. These singularities, which we may interpret as waterfalls or rapids, produce shocks in the flow of water. While both of those mechanisms are present on the convex surfaces that emerge and exist during the early stages of the evolution of the land surface, the second shock producing mechanism dominates the concave surfaces characterizing the mature stage of evolution.

In addition to these sources of noise, small round-off errors in the numerical computations provide a continuous source of small random inputs to both the flow of water over the surface and the erosion of the surface. Since the numerical procedures that underly the current analysis (described in section 3.1) are more accurate than the procedures used in [61], they do not require the suppression of machine noise with the use of large amounts of *artificial viscosity* in order to obtain convergent computations. Instead there are very small amounts of numerical viscosity present in these more accurate methods, enough to suppress round-off errors but small enought to give an accurate presentation of the evolution of the small scales.

In summary, we have identified two mechanisms capable of generating noise in the system, in addition to the initial random perturbations to the surface and the extremely small

numerical noise. While these randomly driven disturbances occur at the short time-scales defined by $\tau = t/\eta^2$ and $\tau = t/\eta$, their cumulative effect is felt in the sediment flow on the long time-scale of erosion through the dependence on h and the term $-\eta\frac{\partial h}{\partial t}$. *These effects constitute the random variations in the model (9)-(10) that drive the dynamics of the system.*

2.5 Complex Surfaces, Structure Functions, and Scaling Relations

Surfaces and interfaces modeled by nonlinear PDEs driven by noise are, in general, too complex to allow an analytic representation. In many cases the solutions that exist do not even possess derivatives. Hence various methods have been developed to represent the structure, and particularly the statistical structure, of such surfaces. An important set of characterizations involve *structure functions* that take the form

$$\rho_k(\mathbf{x}, t) = < |F(\mathbf{x} + \mathbf{y}, t) - F(\mathbf{y}, t)|^k > \tag{14}$$

in which $F(\mathbf{x}, t)$ is some function characterizing a surface, $\mathbf{x} = (x, y)$ are lag variables, $<>$ represents the expectation operator taken over some ensemble of (randomly selected) surfaces, and $k = 1, 2, ...$ is a positive integer. If these structure functions possess a scaling, then we can conclude that the system is *statistically self-similar* and has the same structure and perhaps dynamics on all length scales including arbitrarily small ones.

Certain cases of the structure function are especially important in applications, including the case $k = 1$, in which $\rho_1(\mathbf{x}, t)$ represents the mean value of the differences of the function at different distances apart, and the case $k = 2$, which is often employed in terms of its square root

$$W_F(\mathbf{x}, t) \equiv \rho_2(\mathbf{x}, t)^{\frac{1}{2}} = < |F(\mathbf{x} + \mathbf{y}, t) - F(\mathbf{y}, t)|^2 >^{\frac{1}{2}} \tag{15}$$

to represent the root mean square of the elevation differences as a function of different distances of separation $\mid \mathbf{x} \mid$. This function, which is known variously as the *variogram, height-height correlation function*, *roughness function*, or *width function* [16, 34, 56], is often used to characterize the roughness of the surface in the direction of growth. We may also interpret $W_F(\mathbf{x}, t)$ as measuring the correlation between heights at points separated by a distance $\mid \mathbf{x} \mid$. We note that the width function and the spatial autocorrelation function

$$C_F(\mathbf{x}, t) = < F(\mathbf{y} + \mathbf{x}, t)F(\mathbf{y}, t) > \tag{16}$$

are related by

$$C_F(\mathbf{x}, t) = \sigma_F^2(t) - \frac{1}{2}W_F^2(\mathbf{x}, t), \tag{17}$$

in which σ_F is the variance [9]. An increase in W corresponds to an increase in correlations between two points on a surface. However, the scalings of C and W^2 are, in general, different. Brownian motion, for example, is characterized by $\sigma_F(t) \sim t^{1/2}$ but by $W_F \sim t^{1/4}$.

Important questions concern the manner in which structure functions and their analogs vary with changes in the values of the lag variable $\mathbf{x}$ and time, and in particular, whether their variation may be represented in terms of simple polynomial functions. In many applications, the width function possesses particularly simple asymptotic forms representing *scaling invariances* in the system. It is often the case that the behavior of a surface is well described by asymptotic forms of (15) that are applicable in different space-time regimes of the system's dynamics (see, for example, [56, 38, 55]). Such forms have the representation

$$W_F(\mathbf{x}, t) \sim t^{\beta} f\left(t/|\mathbf{x}|^z\right), \tag{18}$$

in which t is time, $\mathbf{x}$ is the lag variable, β is known as *the dynamic roughness exponent*, and z is called *the dynamic exponent*. An interpretation of these scaling relations is that there is an *equivalence between the temporal and the spatial scales given by* $t \sim |\mathbf{x}|^z$ that allows one to define various space-time regimes of the process.

The roughening of the surface, such as occurs for example when channelized forms emerge on fluvially eroding surfaces, is a transient phenomenon. In the space-time regime of transient behavior the *roughness* of a surface, as measured by $W_F \sim t^{\beta} f(t/\mathbf{x}^z)$, typically increases. This regime is characterized by $f(y) \sim constant$ for $y << 1$, from which it follows that the width function may be approximated by a simple power law

$$W_F(\mathbf{x}, t) \sim t^{\beta} \tag{19}$$

in which β is the dynamic roughness exponent. The length of this transient time interval is of the order $t \sim L^z$, where L is the size of the system [10]. Following this transient period, there may be a space-time regime of stationary, or *saturated*, behavior in which the roughness of the surface fluctuates about statistically steady values. In this regime, which is characterized by $f(y) \sim y^{-\beta},\ y >> 1$, pairs of points on the surface that are separated by a distance $|\mathbf{x}| \sim L$ are characterized by differences in the expected values of the width function of order of $|\mathbf{x}|^{\chi}$. Since the dynamic and *spatial roughness exponents* are related by [11]

$$\chi = z\beta, \tag{20}$$

[9] In practice the variance is always finite for topographic surfaces.

[10] This indicates that the system takes time $t = L^z$ to completely reorganize itself.

[11] This indicates that the dynamic exponent z may be interpreted as the constant of proportionality linking the temporal and spatial scales.

the width function in this case may be approximated by the relationship

$$W_F(\mathbf{x}, t) \sim \mid \mathbf{x} \mid^{\chi} \tag{21}$$

in which χ is *the spatial roughness exponent*. It is this growth in spatial roughness that is easiest to compute and is estimated numerically below. In surface growth models in which $W_F(\mathbf{x}, t)$ measures the roughness of the surface in the direction of the growth, χ may be interpreted as the fractional part of the dimension of the surface, i.e., the embedding or avalanche dimension is $d + \chi$, where $d = 1$ is an integer. Roughly speaking the dimension d characterizes the spatial extent of the avalanche and χ characterizes the height of its "bulge".

If the roughness exponent χ does not depend linearly on k then the scaling is called *multi-fractal*, while if the scaling exponent depends on the direction of the lag vector, the scaling is said to be *anisotropic* (see, for example, [35], [11]). For an arbitrary structure function of order k, for example, this may take the form $\rho_k((x, y)) = constant \times (|x|^{k\chi_x} + |y|^{k\chi_y})$ in which $\chi_x \neq \chi_y$. In the case of single-fractal and isotropic scalings, only two scaling exponents are required to determine all others (see, for example, [33]). The *fractal dimension*, for example, may be represented as $D_f = n - \chi$, where n is the total (spatial) dimension, while the *embedding* or *avalanche* dimension may be represented by $D = d + \chi$, where d is the "dimension" of the interface [12]. The avalanche dimension relates the spatial extent of an avalanche to its temporal growth, $|\mathbf{x}| \sim t^D$ [13].

An important class of statistical phenomena may be characterized in terms of processes that take the form of fractional Brownian motion [40]. The variance of such processes, which are of relevance for the analysis presented in Section 5.3, satisfies scaling laws of the form

$$\sigma \sim t^H f(t/|\mathbf{x}|^z) \tag{22}$$

in which $0 < H < 1$ is known as the Hurst exponent. This exponent is the leading exponent in the rate of increase of the correlation function and is, in general, different from the temporal roughness exponent β that gives the temporal rate of increase of the structure functions.

The usual Brownian motion is a special case of fractional Brownian motion with Hurst exponent $1/2$ and with time increments that are uncorrelated in time. For fractional Brownian motion with Hurst exponent $H > 1/2$, increasing trends in the past lead to a further increases in the future while for fractional Brownian motion with Hurst exponent $H < 1/2$, decreasing trends in the past lead to further decreases in the future for arbitrarily large times.

[12] 1 for a fractal line, 2 for a fractal surface, etc..

[13] We may define an avalanche as a burst of activity in surface growth, in which a significant number of sites move together.

It is important to note that systems characterized by regular Brownian motion achieve a statistically stationary state with respect to their time-increments in the long run, whereas for fractional Brownian motion $H \neq 1/2$, the long-time state is not stationary. However, in both cases the long-time behavior is different from the transient behavior, which explains the difference between the Hurst coefficient and the temporal roughness exponent β. It follows that for Brownian motion ($H = 1/2$) the (generalized) time derivative is statistically stationary *white noise*.

2.6 Estimating the Scaling Exponents from Experimental Data

In the analysis presented below, we employ width functions to characterize the correlation structure of: (1) the *water depth* h in terms of the width function W_h; (2) the *water surface elevation* H in terms of the width function W_H; and (3) the *water surface slope* $\mid \nabla H \mid$ in terms of the width function $W_{|\nabla H|}$. Because of the relation (20), we only require two exponents in characterizing scaling relations involving these width functions and we employ the temporal and spatial roughness exponents β and χ. The spatial roughness exponent is computed from the scaling of the width function (15). We also make use of the fact that we may represent various scaling relations in terms of scaling of the (temporal) power spectrum, see [41]. For present purposes, it suffices to note that

$$\hat{C} \;=\; S(k) \sim \frac{1}{k^{1+2\beta}} \tag{23}$$

in which $S(k)$ represents the power spectrum at frequency k. Employing the power spectrum, we may compute the temporal roughness exponent β independently of the spatial roughness coefficient χ, and thus also obtain the dynamic coefficient $z = \chi/\beta$ using equation (20).

In validating and interpreting our results, we make use of known results concerning the scaling of solutions to equations in the same universality classes as those we examine below.

3 The Two Scalings: Numerical Results

Numerical simulations of the evolution of eroding surfaces generated by equations (9), (10) provide convincing evidence for the emergence of two fundamental landscape scalings. We briefly discuss the numerical methods underlying our analysis and then describe the scaling results.

3.1 Numerical Methods for Accurate Scaling Relations

While the numerical methods employed by Smith, Merchant, and Birnir [61] in solving equations (6), (7) lead to solutions that represent the correct, separable landscapes, they turn out to be insufficient for deriving numerically accurate scaling results. These solution methods, described in detail in [61], involve alternative computations of equilibrium water flows and very small erosional changes to the land surface, using an explicit, two-step MacCormack predictor-corrector finite-difference scheme adapted for two-dimensional overland flow [64, 60]. This scheme involves the use of artificial viscosity to avoid numerical oscillations [30]. By suppressing the noise that is amplified by the instability of the initial surface, however the *small scale dynamics* become inaccurate. Hence these methods are inadequate for numerically accurate estimates of the scaling parameters.

For the purposes of the current investigation, we designed a significantly more accurate numerical scheme capable of representing the correct small scale dynamics and based on the existence theory of the nonlinear PDEs. By solving the hyperbolic equation (9) in two dimensions using an *upwind scheme*, we render the scheme stable and eliminate the need for artificial viscosity.[14] A Crank-Nicholson scheme may then be applied in computing the nonlinear gradient flow (10) and by solving the resulting implicit matrix equation. This computation is significantly less expensive than the previous computations because in the explicit scheme significant amounts of computation are required in order to dampen instabilities in the water depth by artificial viscosity [15].

We note in passing that the comparison with the previous numerical scheme is an interesting study in numerical analysis and we refer the reader to [10] for further details on the new numerical scheme. This code will also be made public.

3.2 The Numerical Simulations

Employing the new code, we ran a full suite of computational experiments simulating the erosion of a section of a linear ridge satisfying the initial and boundary conditions (12), (11), which are the same as those employed in [58, 61]. These experiments were run using various refinements of the grids and over all relevant parameter values of the model (9, 10) to provide a basis for deriving the scaling relationships that we describe in section 3.3.

In Figure 1-4, we show typical configurations of eroding surfaces, water depth, and slopes

[14] An upwind scheme still produces small amounts of numerical viscosity but of magnitude less than the artificial viscosity in [30].

[15] We implemented the scheme on a parallel supercomputer at the University of California, Santa Barbara, employing as a final step the implementation of a *four color scheme* to parallelize the procedure for maximum computational advantage.

of the water surface for values of the sediment transport parameters $\gamma = 2, \delta = 2$ at two characteristic times: when 10% of the surface is eroded and when 60% of the surface is eroded, which is well into the separable regime. In these figures, which indicate that the $x-axis$ runs laterally across the ridge and the $y-axis$ runs longitudinally down the ridge, we represent the water surface $H = z + h$, in Figures 1 and 3; we represent the water depth h and the gradient of the water surface $\mid \nabla H \mid$, in Figures 2 and 4, again at 10% and 60% of the surface eroded.

3.3 The Basic Scaling Results

The scaling results that we derive in this section are based on numerical solutions obtained from experiments with sediment transport parameters $\gamma = 2, \delta = 2$. We note that there is no significant variation of our results in neighborhoods of such values and that experiments starting with different initial conditions result in the same scaling results.

Scaling results were obtained by computing the width function (21) over sections of the ridge either in the x-direction transverse to the ridge or in the y-direction parallel to the longitudinal axis of the ridge. Estimates of the exponent χ of the asymptotic forms (21) were provided by the slope of log-log plots of width functions averaged over an ensemble of results from randomly seeded runs. Numerical estimates of χ were obtained by least squares fits. In Figures 5 and 6, we illustrate typical log-log plots taken at a given transverse section of the ridge for the water depth h and water surface H respectively. The R^2-value associated with each plot provides a measure of the appropriateness of scaling relation.

It is convenient to view the values of the scaling parameters as a function of their location on the ridge. For example, the values characterizing scalings in directions transverse to the ridge may be viewed as a function of the location on the longitudinal axis of the ridge $\chi_x = \chi_x(y)$. We may then represent the scaling relations found over sections of the ridge by plotting both the estimated value of χ and the associated R^2-value as a function of location on the ridge. This is illustrated in Figures 7-10, in which we represent of the scalings for h, H, and $\mid \nabla H \mid$, at different cross-sections of the ridge at the two erosion times.

The scaling results from solutions to (9) and (10) for a time of 10% eroded are shown in Figures 7-8 as a function of distance upslope. These figures indicate that, initially, the structure functions of the water depth (W_h) and the gradient of the water surface ($W_{|\nabla H|}$) scale with the roughness exponent $\chi_x = 0.5$ in the lateral (or x) direction. We note that this value characterizes a large range of y-locations on the ridge. It should be noted that, while in Figure 8 the *numerically computed* value of the exponent of $|\nabla H|$ is $\chi = 0.25$, the *true* value of this exponent is 0.5. The apparent discrepancy is caused by the initial slope $H_y = c$ whose cross-term appears when one squares the components of $|\nabla H|$ and then

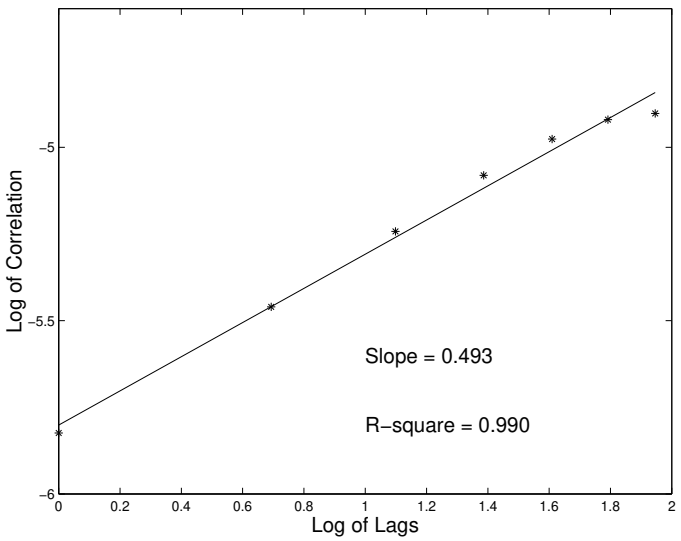

Figure 5: Log-log plot of water depth at a given cross-section

takes the square root, see Appendix B.

As the solutions converge towards the separable forms defining the mature landscape (8), the above scaling disappears to be replaced by another scaling. In a neighborhood of the separable landscape, at 60% of the surface eroded, the water surface scales in the x-direction as shown in Figure 9 with the scaling exponent $\chi_x = 0.7$, again over a large range of y-locations on the ridge. While the corresponding *numerically computed* scaling of $|\nabla H|$ is $\chi_x = 0.35$, we note again that the *true* value of this exponent is 0.7 rather than the smaller value, for the reasons stated above and explained in Appendix B.

We note, therefore, that at 10% of the landsurface eroded the water depth h and the gradient of the water surface $|\nabla H|$ scale, whereas at 60% of the landsurface eroded the water surface H itself and the gradient of the water surface $|\nabla H|$ show the same scaling. In both cases $|\nabla H|$ controls the scaling; in the former case it controls h; in the latter case it controls H. Our results indicate that H does not scale over any significant range of y-values in the first case while h has a different scaling in the latter case. This clearly indicates that two distinct processes lead to the two different scalings.

In characterizing some of the details of these scalings, we may interpret the longitudinal axis of the ridge as a time scale and may think of the figures as representing increasing time values. Figures 3-6 indicate that the scaling is uniform neither in y nor in time. Initially, or at 10% of the landsurface eroded, the roughness exponent of W_h is approximately constant (0.5) for most of the upslope direction but deviates at the top where the boundary exerts its influence, as shown in Figure 7. As time increases this plateau disappears and one obtains

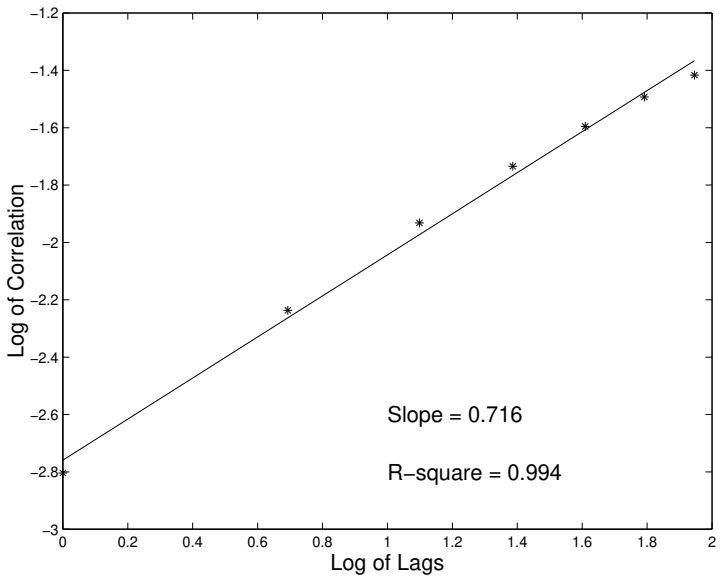

Figure 6: Log-log plot of the water surface at a given cross-section

different exponents as functions of y. Similarly, the width function of the water surface W_H initially shows different values of χ_x as a function of y but as the solution approaches the separable surface, a plateau is formed where the roughness exponent is constant (0.7) for a large portion of the upslope directions, see Figure 9. Thus the existence of a plateau in χ_x, as a function of y, implies that the ensemble average possesses a scaling. These figures show the slope (or scaling exponent) as a function of the distance y up the ridge.

The scaling is anisotropic in the sense that there is no plateau in χ_y as a function of x, but this is more due to the fact that the upslope direction is really a time-scale, rather than a real anisotropy. We examined the slope-free part of the gradient H_x, see Appendix C, in various directions but found little evidence of significant anisotropy. We also examined both higher and lower structure functions $\rho_k,\ k \neq 2$, or moments, of h, H and ∇H but found little evidence that either scaling was multifractal. The spatial scalings were checked by computing the spatial power spectrum. The temporal power spectrum was very noisy due to the unevenness of the temporal evolution in the upslope direction and machine noise. It only gave reasonable results for the first scaling and then only for the water depth h.

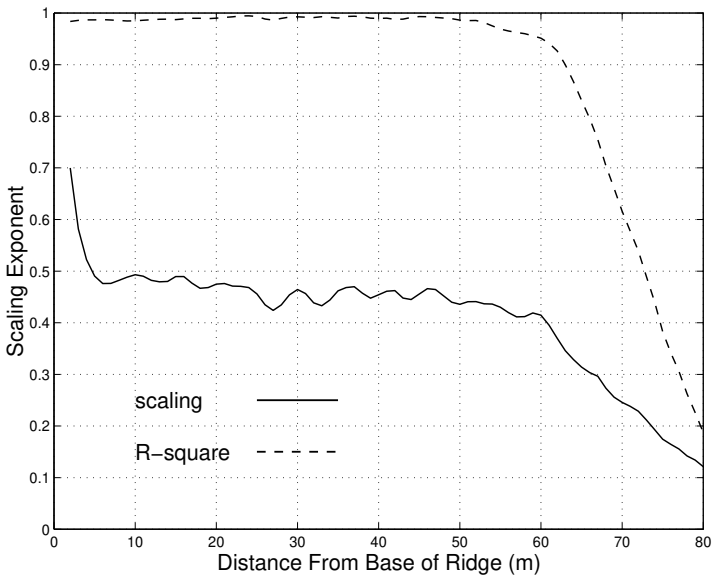

Figure 7: The scaling for the water depth

4 The Emergence of Channels and the First Fundamental Scaling of Fluvial Landscapes

We now consider the emergence, early in the evolution of surfaces determined by the model (9)-(10), of scalings in the water depth h and in the slope of the water surface $\mid \nabla H \mid$ that are associated with the emergence of channelized flows. In particular, we focus on explaining how the fundamental scalings of 0.5 in the width functions W_h and $W_{|\nabla H|}$ are driven by noise generated by instabilities in the water flow. Since this process is intimately related to the emergence of channels, we view the nature and origin of this fundamental diffusive scaling as providing significant insight into the manner in which the early growth of channel-like features is dominated by random influences. In particular, these considerations provide a model of how random influences determine the location of channels and, as such, may be used in justifying the assumptions that underly the applications of various discrete stochastic modeling approaches to fluvial networks.

4.1 The Emergence of Channels and the Fundamental Channelization Instability

Results from previous analyses of models of the form (1), (2), provide insight into the earliest stages of channel evolution. Smith and Bretherton [59] analyzed versions of equations (6), (7) in which the constitutive relation for sediment transport was generalized to

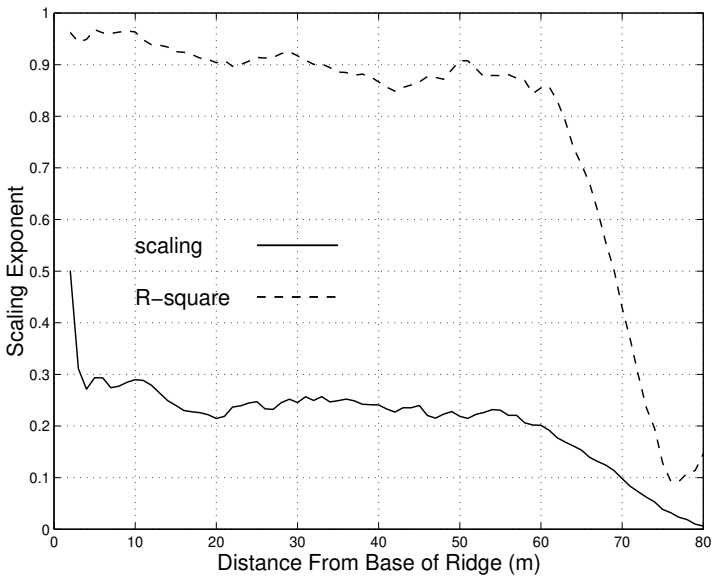

Figure 8: The first scaling of the gradient of the water surface

$q_s = F(q_w, \mid \nabla H \mid)$, with $\frac{\partial F}{\partial q_w} > 0$ and $\frac{\partial F}{\partial S} > 0$ [16]. Based on a linearized analysis of these equations, they showed that if $\frac{F}{q_w} < \frac{\partial F}{\partial q_w}$, then small amplitude disturbances of small lateral scale will initially grow very rapidly, with the initial growth rate being greatest for disturbances with the smallest wavelengths. This criterion is equivalent to the condition $\gamma > 1$ for the particular sediment transport law embedded in equation (7). This latter fact leads to the particularly simple criterion that channel incision takes place when the sediment transporting capacity of water increases at a faster rate than the discharge of water, and occurs faster for shorter wavelength disturbances because the convergence of water flows is greater for disturbances with steeper lateral gradients. These results were reconfirmed by Loewenherz [15], who also derived the full asymptotic solution in the small wavelength limit.

The simulations of Smith, Merchant, and Birnir [61] confirmed numerically that the highest initial growth rates occur at the highest frequencies. They also found, however, that the growth rates of lower frequencies became dominant after a very short initial period, presumably as a result of saturation of the nonlinearities in the erosion process. The significance of these results for the present analysis is that they show that the instabilities of the initial erosion process are highly susceptible to being driven by random influences.

We now provide an alternative analysis of these early channel forming instabilities of (1), (2) which provides additional insight into the underlying processes. In particular, we linearize the equations (9) and (10) about an initial surface, linear in y and uniform in the x

[16] The term $\frac{\partial h}{\partial t}$ in equation (7) was not considered in these analyses.

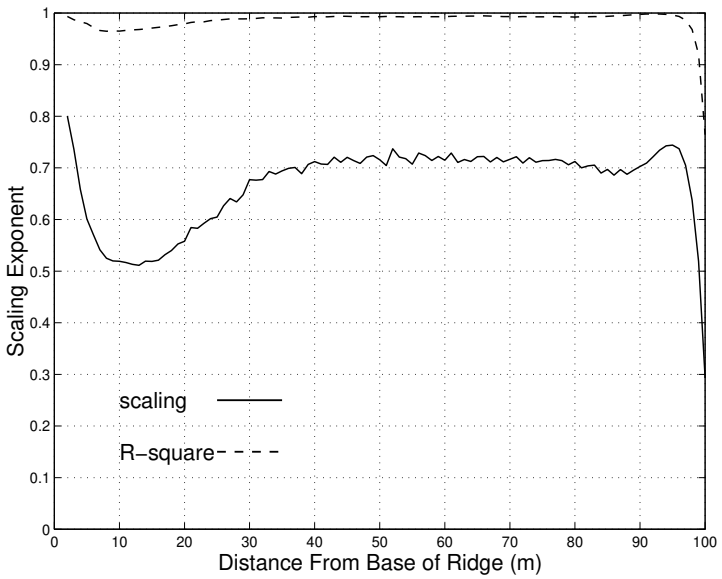

Figure 9: The scaling of the water surface

direction, and a constant water depth $H = cy, h = d$ [17], where c and d are constants. We note that, while these initial conditions differ from those in the stability analyses of [59, 15], they lead to essentially the same results but can be regarded as one step earlier in the evolution from a ridge with constant slope. As such they provide an additional interpretation of the channel forming instabilities.

We let

$$h = d + \epsilon a(x, y, t), \quad H = cy + \epsilon b(x, y, t),$$

and include a (very) small stochastic variation in the rainfall rate

$$R = R_o + \epsilon r(x, y, t).$$

The linearized system of equations becomes

$$\eta^2 \frac{\partial a}{\partial t} = \frac{5}{3} d^{\frac{2}{3}} c^{\frac{1}{2}} \frac{\partial a}{\partial y} + \frac{d^{\frac{5}{3}}}{c^{\frac{1}{2}}} \left(\frac{\partial^2 b}{\partial x^2} + \frac{1}{2} \frac{\partial^2 b}{\partial y^2} \right) + r, \tag{24}$$

$$\begin{aligned} \frac{\partial b}{\partial t} = \frac{5}{3} d^{\frac{2}{3}} c^{\frac{1}{2}} \left(\frac{1}{\eta} + 2 c^{\frac{5}{2}} d^{\frac{2}{3}} \right) \frac{\partial a}{\partial y} &+ \frac{d^{\frac{5}{3}}}{c^{\frac{1}{2}}} \left(\frac{1}{\eta} + c^{\frac{5}{2}} d^{\frac{5}{3}} \right) \frac{\partial^2 b}{\partial x^2} \\ &+ \frac{d^{\frac{5}{3}}}{c^{\frac{1}{2}}} \left(\frac{1}{2\eta} + \frac{5}{2} c^{\frac{5}{2}} d^{\frac{5}{3}} \right) \frac{\partial^2 b}{\partial y^2} + \frac{r}{\eta}. \end{aligned} \tag{25}$$

[17] Strictly speaking we should let the water depth go to zero at the top of the ridge to satisfy the boundary conditions there. This, however, makes no difference to the results of the analysis.

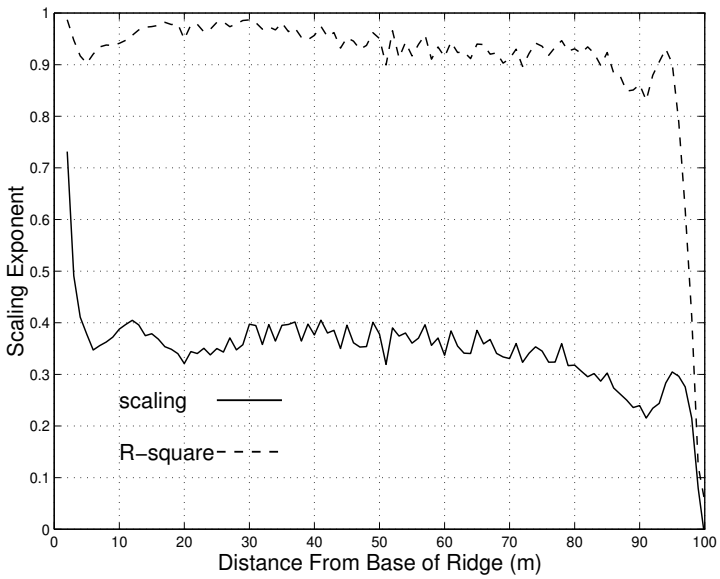

Figure 10: The second scaling of the gradient of the water surface

The numerical results reported in Section 3.3 indicate that the variation of the water surface b does not scale, initially at least, with a roughness exponent that is constant over a range of y values. However, the gradient ∇b does scale from which it follows that the equations generating the scaling must be equation (24) and [18]

$$\begin{aligned}\frac{\partial \nabla b}{\partial t} &= \frac{5}{3} d^{\frac{2}{3}} c^{\frac{1}{2}} \left(\frac{1}{\eta} + 2c^{\frac{5}{2}} d^{\frac{2}{3}}\right) \frac{\partial \nabla a}{\partial y} + \frac{d^{\frac{5}{3}}}{c^{\frac{1}{2}}} \left(\frac{1}{\eta} + c^{\frac{5}{2}} d^{\frac{5}{3}}\right) \frac{\partial^2 \nabla b}{\partial x^2} \\ &\quad + \frac{d^{\frac{5}{3}}}{c^{\frac{1}{2}}} \left(\frac{1}{2\eta} + \frac{5}{2} c^{\frac{5}{2}} d^{\frac{5}{3}}\right) \frac{\partial^2 \nabla b}{\partial y^2} + \frac{\nabla r}{\eta}. \end{aligned} \tag{26}$$

Since $\eta = d/100c \sim h/H$ is small, the term $\nabla r/\eta$ can be large if r is not exceedingly small. However in most cases r can be taken to be very small and therefore we have dropped it in the equation below, still keeping in mind that very small fluctuations are always present in landscape evolution.

A straight-forward stability analysis shows that the initial unchanneled surface is unstable on the long time-scale of the sediment flow. We set $\frac{\partial h}{\partial t} = 0$ in the equations (9) and (10), or equivalently use the slow equations (6) and (7), with $\frac{\partial h}{\partial t} = 0$. On taking the gradient, equation (24) becomes

$$\frac{5}{3} d^{\frac{2}{3}} c^{\frac{1}{2}} \frac{\partial \nabla a}{\partial y} = -\frac{d^{\frac{5}{3}}}{c^{\frac{1}{2}}} \left(\frac{\partial^2 \nabla b}{\partial x^2} + \frac{1}{2} \frac{\partial^2 \nabla b}{\partial y^2}\right) \tag{27}$$

[18] We have merely applied the ∇ operator to equation 25.

and a substitution into (26) gives the equation

$$\frac{\partial \nabla b}{\partial t} \;=\; \frac{5}{3} d^{\frac{2}{3}} c^{\frac{1}{2}} [(-2+d)\frac{\partial^2 \nabla b}{\partial x^2} + (-\frac{1}{2}+3d)\frac{\partial^2 \nabla b}{\partial y^2}], \tag{28}$$

which is a negative heat equation with $d < 1/6$. Recall that d is the initial water depth that can be taken to be very small. The dispersion relation for (28)

$$\omega \;=\; \frac{5}{3} d^{\frac{2}{3}} c^{\frac{1}{2}} [(2-d)k_1^2 + (\frac{1}{2}-3d)k_2^2],$$

shows that all the spatial frequencies are unstable and that the smallest frequencies grow the fastest. We note that this is essentially the same instability as discovered by [59, 15], but expressed in terms of the unstable growth of arbitrarily small (spatial) frequencies in the slope of the surface rather than in the elevation of the surface. In particular, this result is nicely in accordance with an interpretation of the stability results of Smith and Bretherton [59] that suggests that smaller perturbations grow at faster rates because, for a given amplitude of perturbation, they possess steeper slopes which lead to more rapid convergence of eroding water flows.

The significance of this result follows from our preceding conclusion that fluctuations of arbitrarily small frequencies are always present, and that these instabilities always lead to the exponential growth of slopes, and the consequent channelization of the initial surface, over the associated short time-scale described in Section 3.

4.2 The Generation of Noise During Initial Channelization and Shocks in the Water Surface

We now provide and justify a hypothesis that shocks in the water surface provide a significant source of white noise that drives the erosion process. In particular, we show that shocks form in the flow of water over initially *convex* portions of the ridges between emerging channels. Although a concavity in the longitudinal profile emerges early near the base of the ridge, moves longitudinally towards the top of the ridge, and then expands laterally across the emerging valleys (for an analysis of this process, see [58]), the lateral profiles of the ridges between emerging channels take an initial form that is typically convex in their upper parts. Two upridge sequences of lateral profiles that indicate, respectively, the existence of initially convex sections of the interfluvial ridges at 10% of the surface eroded and the dominance of fully concave lateral profiles at 60% of the surface eroded are shown in Figure 11.

We base our analysis of the mechanism that provides a significant source of white noise on these convex sections of ridge by linearizing equations (9) and (10) around a convex

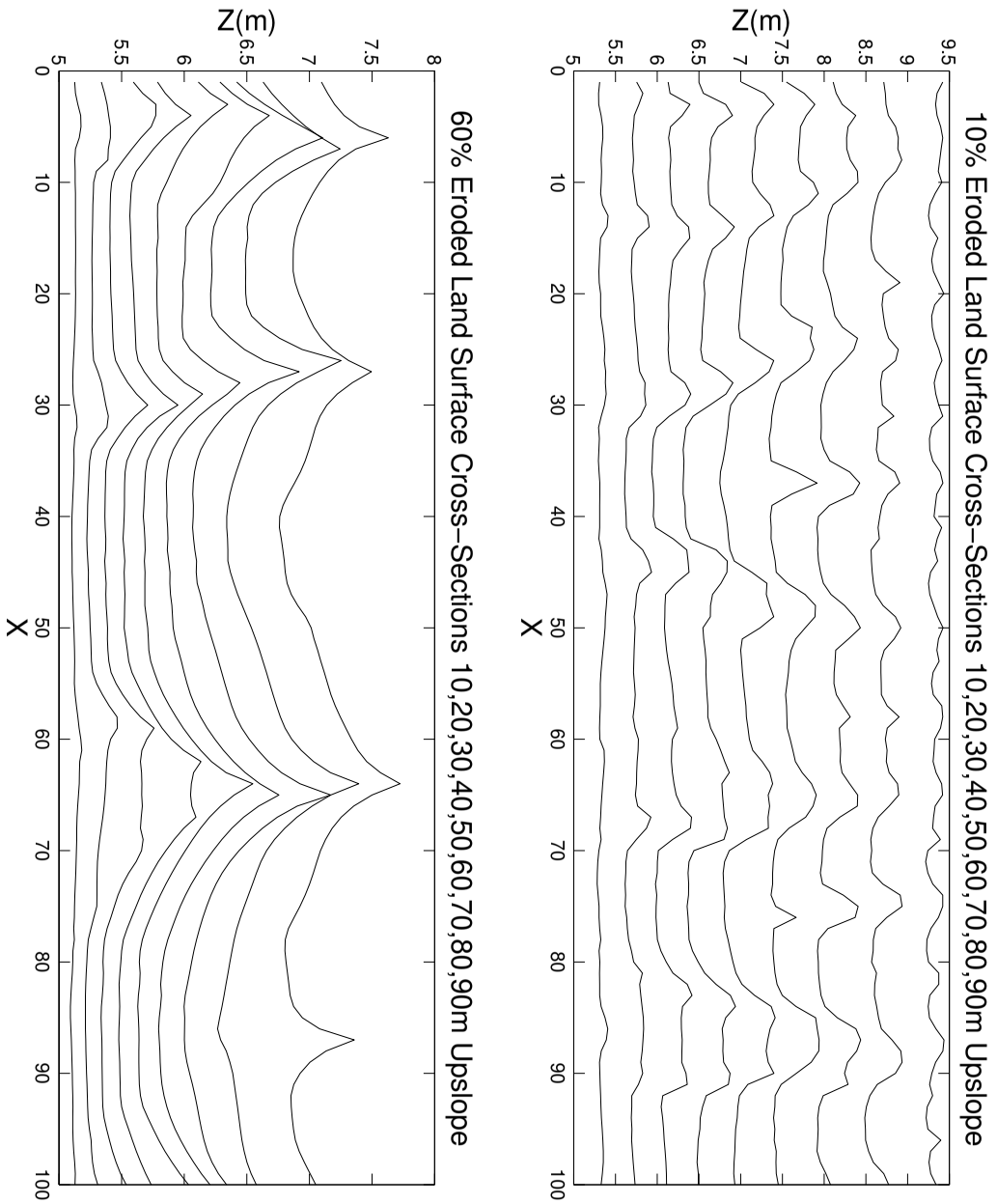

Figure 11: Transverse Sections at 10m separation on the Eroding Ridges at 10% and 60% Eroded

(interfluvial) profile. In particular, we assume that our perturbed solutions take the form

$$h(x,y,t) = h_1(x,y,t) + v(x,y,t), \;\; H(x,y,t) = H_1(x,y,t) + u(x,y,t)$$

where $H_1(x,y,t)$ represents a convex portion of an interfluvial ridge, $h_1(x,y,t)$ the depth of water flow over this portion, and $v(x,y,t), u(x,y,t)$ are the small perturbations to these quantities. The linearized equations are

$$\eta^2 \frac{\partial v}{\partial t} \;=\; \nabla \cdot \left[\frac{5}{3} h_1^{\frac{2}{3}} \frac{\nabla H_1}{|\nabla H_1|^{1/2}} v \right] \;+\; \nabla \cdot \left[h_1^{\frac{5}{3}} \frac{\nabla u}{|\nabla H_1|^{1/2}} - \frac{1}{2} h_1^{\frac{5}{3}} (\nabla H_1 \cdot \nabla u) \frac{\nabla H_1}{|\nabla H_1|^{3/2}} \right], \quad (29)$$

$$\begin{aligned} \frac{\partial u}{\partial t} \;&=\; \nabla \cdot \left[\frac{h_1^{\frac{5}{3}}}{|\nabla H_1|^{1/2}} (\frac{1}{\eta} + h_1^{\frac{5}{3}} |\nabla H_1|^{5/2}) \nabla u \right] \;+\; \nabla \cdot \left[\frac{5}{3} h_1^{\frac{2}{3}} \frac{\nabla H_1}{|\nabla H_1|^{1/2}} (\frac{1}{\eta} + 2 h_1^{\frac{2}{3}} |\nabla H_1|^{5/2}) v \right] \\ &+\; \nabla \cdot \left[h_1^{\frac{5}{3}} (\frac{-1}{2\eta} + 2 h_1^{\frac{5}{3}} |\nabla H_1|^{5/2}) (\nabla H_1 \cdot \nabla u) \frac{\nabla H_1}{|\nabla H_1|^{3/2}} \right], \end{aligned} \quad (30)$$

The first equation is a hyperbolic equation for v driven by ∇u. The second equation is a parabolic equation for u driven by v and ∇v.

Because the flow of water occurs down the gradient of the surface $H_1(x,y,t)$, we may view the first equation as a hyperbolic PDE in one space dimension. Namely, if we let $\mathbf{u}$ denote the unit vector in the direction of the gradient of the water surface, we can write the first equation in the form

$$\eta^2 \frac{\partial v}{\partial t} \;=\; \frac{5}{3} h_1^{\frac{2}{3}} |\nabla H_1|^{1/2} \mathbf{u} \cdot \nabla v \;+\; g(x,y,t) v \;+\; f(x,y,t). \quad (31)$$

If we then let the scalar s parametrize the direction of the gradient, we may rewrite this equation in the form

$$\eta^2 \frac{\partial v}{\partial t} \;=\; \frac{5}{3} h_1^{\frac{2}{3}} |\nabla H_1|^{1/2} \frac{\partial v}{\partial s} \;+\; g(x,y,t) v \;+\; f(x,y,t). \quad (32)$$

The last equation is analyzed in Appendix B, where it is shown that *it develops shocks if the profiles about which we linearize are convex* and have knick-point singularities.

After operating with the gradient, we may lump the terms in (30) causing the instability and involving the water depth together into a single noise term r. This term is the noise generated by the shocks in (31) which is no longer small, as are the initial perturbations.

Specifically, as we show in Appendix B, this noise takes the form of white noise. We obtain, as a result, a heat equation driven by white noise,

$$\frac{\partial(\nabla u)}{\partial t} \;=\; \nu\Delta(\nabla u) \;+\; r(x,y,t), \tag{33}$$

where $\nu = \frac{d^{\frac{5}{3}}}{\eta c^{\frac{1}{2}}}$. The solution to this equation is well-known to be Brownian motion

$$\nabla u = \int_0^t \int_{\mathbb{R}^2} \frac{exp(-|\mathbf{x}-\mathbf{y}|^2/4\nu(t-\tau))}{(4\pi(t-\tau))}\, dW(y,\tau), \tag{34}$$

where $\mathbf{x} = (x,y)$, $\mathbf{y} = (x',y')$ and $dy = dx'dy'$ is the volume element in $\mathbb{R}^2$ and formally the white noise can be understood as $dW(y,\tau) = r(\mathbf{y},\tau)dyd\tau$. Adding this solution to $(0,c)$ we obtain *a random walk*

$$\nabla H \;= (0,c) + \epsilon\nabla u,$$

of the slopes of the water surface up the channelizing surface, driven by white noise. The water depth follows suit, by equation (9).

4.3 The Scaling of the Channelization Process

We are now in a position to interpret the scaling exponents that emerge during the initial stages of the erosion process by using well-known results from the theory of the evolution of complex interfaces. We note that equation (33) is the Edward-Wilkinson equation of interface growth theory, which is known to model surfaces over which the deposition rate equals the removal (or evaporation) rate (see [33]). The values of the exponents of the associated width functions are

$$\beta = \frac{1}{4}, \;\; \chi = \frac{1}{2}, \;\; z = 2.$$

We note that β is different from the Hurst exponent, which has the value $1/2$ for a pure random walk, because β measures the transient towards a pure random walk. The resulting motion may be interpreted as a random walk occurring on top of another random walk. The gradients of the water surface walk randomly in the upslope direction and the nodes where $\nabla H(x+y) - \nabla H(y)$ vanishes do another random walk on top of that. Hence *the configuration of the emerging channel forms may be characterized in terms of random walks.* We note that this important finding provides a physically-based justification for various

stochastic modeling approaches to the emergence of network forms such, for example, as described in [6, 27].

This surface sits in a 3-dimensional space and the fractal dimension is $D_f = 3 - \chi = 5/2$. The avalanche dimension is the same $D = 2+\chi = 5/2$. The temporal roughness coefficient β may be obtained from the power spectrum. The reason is that the erosion is initially greatest towards the bottom of the slope and then the channelization migrates upstream and ends when the concavity, forced by the boundary conditions at the bottom of the slope, reaches the top of the slope, see [58, 61]. This implies that the y direction is really a time-scale. That is, instead of measuring the percentage of the landsurface that has been eroded, we can simply measure how far upstream the channelization process has migrated.

5 Self-Similar Landscapes and the Second Fundamental Scaling of Fluvial Landscapes

In further support of our hypothesis that the observed scalings arise from identifiable physical processes, we now show that the second scaling of 0.7 may be interpreted as characterizing the erosion processes in the mature, separable landscapes. These processes are driven by noise that is generated by shocks in the water flow over the concave slopes of landscapes such as those shown in Figure 11, and as explained in Appendix B. These shocks are initiated by the singularities in the derivatives of the surface at the top of the concave ridges, see Figure 11. In particular, we provide evidence indicating that this behavior is characteristic of a SOC system called *directed percolation* [33] which describes a number of nonlinear systems including a Reggion field theory [26], [29].

5.1 The Emergence of Self-Similar Landscapes

Smith, Birnir, and Merchant [58] showed that the model (6)-(7) possesses separable solutions $H(x, y, t) = H_0(x, y)T_H(t)$ and $q_w(x, y, t) = q_0(x, y)$. These are associated with an optimality principle by which *mature* eroding surfaces evolve to minimize a simple function of the sediment flux over the surface, subject to two constraints. In their analysis of numerical solutions of (6), (7), Smith, Merchant, and Birnir [61] showed that after a characteristic erosion time [19] the surfaces exhibit clear and convincing evidence of convergence towards such separable solutions, and the corresponding satisfaction of the optimality criterion.

[19] The characteristic time is 45% of the surface eroded for planar initial surfaces with $\gamma = \delta = 2$.

As shown in Section 3, this stage of development is characterized by a dominant scaling of the width function with exponent 0.7. We now analyze the model (9), (10) in order to determine the origin and physical basis for this scaling.

5.2 The Origin of the Second Diffusive Scaling

The scaling of the water flow down the slope of a mature landscape and the resulting scaling of the water (and land) surface is different from the scaling of channel formation in Section 4. To find and analyze this scaling we linearize equations (9) and (10) about the separable solutions (8) representing the mature landscapes of valleys and ridges [20]

$$H = (H_o(x,y)T(t) + \epsilon u(x,y,t)), \;\; h = (h_o(x,y)T^{-\frac{3}{10}}(t) + \epsilon v(x,y,t)),$$

where $H_2 = H_o(x,y)T(t), \;\; h_2 = h_2(x,y)T^{-\frac{3}{10}}(t)$, are the separable solutions of the equations (9, 10). The form of the equations that we obtain for $u(x,y,t), \; v(x,y,t)$ by this linearization process is the same as that characterizing the early period of channel emergence discussed in Section 4.2, namely (29) and (30), but with the terms $H_1(x,y,t), \;\; h_1(x,y,t)$ replaced by the separable solutions $H_2 = H_o(x,y)T(t), \;\; h_2 = h_2(x,y)T(t)$. Again, the first equation is a hyperbolic equation for v driven by ∇u and the second is a parabolic equation for u driven by v and ∇v. Since the first equation is really a hyperbolic PDE in one space dimension, exactly as in Section 4.2, we let the scalar s parametrize the direction of the gradient to obtain

$$\eta^2 \frac{\partial v}{\partial t} \;=\; \frac{5}{3} h_2^{\frac{5}{3}} |\nabla H_2|^{1/2} \frac{\partial v}{\partial s} \;+\; g(x,y,t)v \;+\; f(u,t). \tag{35}$$

A straight-forward analysis of this equation, presented in Appendix B, shows that its solutions develop shocks, for separable surfaces, since concave slopes are the dominant feature of the mature separable landscape, as illustrated in Figure 11.

Thus the water flow develops shocks at the top as it flows down the separable ridges, which then propagate down the slopes. We note that equation (30) now becomes a heat equation driven by noise generated by shocks. However, it is different from equation (33) because the noise is different since the noise term

$$\frac{5}{3} h_2^{\frac{2}{3}} \frac{\nabla H_2}{|\nabla H_2|^{1/2}} (\frac{1}{\eta} + 2h_2^{\frac{2}{3}} |\nabla H_2|^{5/2}) \cdot \nabla v$$

[20] As in our previous analyses, we employ sediment transport parameter values $\gamma = \delta = 2$, noting that similar results hold for parameter values in a neighborhood of these.

is not spatially homogeneous. The water depth h_2 becomes very small at various locations on the separable surface, and at these locations the noise is quenched. One can think of these spots as pinning the surface and creating small local minima in the sediment flow. Thus equation (29), with the separable solutions $H_2(x,y,t),\ h_2(x,y,t)$ replacing $H_1(x,y,t),\ h_1(x,y,t)$, is a diffusion equation driven by *quenched noise*. The other term on the left hand of (30) side is the forcing term that performs the pinning of the surface at the sites where the noise is small. It is known [33], [37], that such processes produce polynomial scaling with the spatial roughness exponent close to 0.7. This implies that the water surface

$$H = H_o(x,y)T(t) + \epsilon u(x,y,t)$$

executes a random motion driven by the derivative of shocks $\nabla h_o + \epsilon \nabla v(x,y,t)$ propagating down the separable surface, being pinned at randomly distributed sites.

5.3 The Scaling of Mature Landscapes

We compute the exponents using the formulas in Section 2.5, starting with the numerical value of the spatial and temporal roughness exponents from Section 3. The values of the exponents are

$$\beta = 0.4,\ \ \chi = 0.7,\ \ z = 1.8.$$

The fractal dimension becomes $D_f = 3 - \chi = 2.3$. The avalanche dimension is $D = 2 + \chi = 2.7$. The values of these exponents are close to the corresponding values for directed percolation in 1+1 dimensions, see [33],

$$\beta = 0.39,\ \ \chi = 0.63,\ \ z = 1.62.$$

The value of spatial roughness coefficient is however larger (0.7) for the mature landscape evolution, than for directed percolation (0.63) because the former process is driven by *quenched noise*. Directed percolation describes a branching process for a density of sites that are alive on a surface and is qualitatively similar to our model (30) of diffusion of the water surface driven by quenched noise. The idea is that the surface gets pinned at random sites, for example, by the water depth going to zero so $H = z$. This clearly happens, for example, at the tops of ridges on mature surfaces (see Figure 4 and [58, 61].) At these sites no sediment transport takes place and the overall landscape evolution is affected. It is, in fact, well-known that landscapes generated by discrete landscape models do not reproduce realistic landscape unless random pinning that models quenched noise is included in the model, see [17]. The latter models also give rise to values of the scaling exponent β similar to ours. There are two remarkable features of the continuous model (9 ,10). First, it generates the noise from the water flowing down the separable surface (8) and shocking as it

passes over surface singularities. Second, it produces the pinning of the surface at random sites where these shocks are absent.

We note with interest that directed percolation is known to be an example of an SOC system [21]. Hence we may interpret the landscapes described by our advective model as SOC systems. The spatial roughness of another well-known SOC model, the nonlinear interface model [33], is 2/3 in one dimension, which is greater that the exponent 1/2 of the Edward-Wilkinson [16] model and our rill evolution model in Section 4. However, the roughness exponent of the evolution of the water surface is even greater, due to the quenching of the noise.

Directed percolation is also believed to describe a Reggion field theory [26], [29] and it has been proposed by Pomeau as describing the injection of vorticity into laminar fluid flow by the boundary. The scaling exponents are obtained by renormalization of the Langevin equation

$$\frac{\partial w}{\partial t} \;=\; \Delta w - \lambda w^2 + F + r(w,t), \tag{36}$$

where F is a pinning force, r is the quenched noise, and λ is a parameter. This equation has a similar form to the gradient of our equation (30). They are both gradient flows that are driven by a pinning force and quenched random noise. However, (30) is a linear equation whereas (36) has a quadratic nonlinear term. It is a very challenging mathematical problem to show that this renormalization can be carried out for the *full* nonlinear system (9 ,10), so as to obtain (36) in the limit.

The value of the dynamic exponent comes out to be $z = 1/D$, where D is the avalanche dimension, see [33]. In directed percolation (DP) all the exponents are determined by χ and $z = 1/(1+\chi)$ by the above relationship, see [33]. Thus in their mature forms, river networks are DP networks being formed on the background of randomly meandering rivers. However, the relationship between z and D does not hold for the channelization process, which is the underlying random walk in Section 4, so the theory of DP does not apply to the initial formation of the network. During this initial stage the river simply execute a pure random walk.

[21] As noted in [33]:

> A system is a SOC system if the scaling (of the structure) functions during the saturated stage is different from the scaling during the transient stage and the activity during the saturated stage occurs in bursts that are self-organized during the transient.

6 Discussion and Significance of the Results

The preceding results on the origins of scaling in solutions to the models (9)-(10) have significant implications both for the evolution of many of the empirical regularities observed in fluvial landscapes and for the manner in which one may model such phenomena. We now describe a few of the more important implications.

6.1 The Origin of Observed Landscape Width Scalings

We note with great interest that empirical results for the height-height correlation function from Ethiopia, Somalia, and Saudi Arabia, as well as values generated for synthetic landscapes as reported by Weissel and Pratson [32], fall in the range 0.5-0.7. The Saudi Arabian data is virtually isotropic with the value 0.5, whereas some of the Somalian data is anisotropic, with values roughly 0.5 in one direction and 0.7 in the other. Although empirical scaling results of this form tend to cluster around a relatively small set of values, these values provide some validation for our theory particularly since our models may be interpreted as representing advective, transport-limited (or *badland*) conditions, which presumably hold in these areas.

The theory presented in this paper suggests that the spatial roughness scaling exponents for landscapes that are approximately described by our advection models should lie in a neighborhood of 0.5 and 0.7. Young channelizing surfaces should exhibit isotropic scaling with exponent close to 0.5, whereas more mature surfaces should be anisotropic. This anisotropic scaling should have the scaling exponent close 0.7 on the slopes of the separable ridges but on channelizing surfaces along the main rivers the scaling exponent should be close to 0.5.

6.2 The Origin of Hack's Law:

The preceding results allow us to derive some of the fundamental scaling results that are known to characterize fluvial landsurfaces. In particular, the avalanche dimension derived in Section 5.3 allows us to derive Hack's Law relating the length of a river l to the area A of the basin that it drains. This area is area of the river network that is given by the avalanche dimensions

$$A \sim l^D$$

and the avalanche dimensions is $D = 1 + \chi$. This relation says that if the length of the main river is l then the width of the basin in the direction, perpendicular to the main river, is l^χ. Stable scalings for the surface emerge together with the emergence of the separable

solutions. We note that in this case $\chi = 0.7$, hence we obtain

$$l \sim A^{\frac{1}{1+\chi}} \tag{37}$$

$$\approx A^{0.6} \tag{38}$$

a number that is in excellent agreement with observed values of the exponent of Hack's law (see, for example, [28]). We note, in particular, that χ is not the roughness coefficient of a fractal Brownian motion. We also note that the area A is the same as the area of voids (areas without branches) in a DP network (see [33]) and that Hack's Law is just the relation between the area of the voids and the length of the DP backbone. Thus one can understand the "oval shape" of most river basins, see [28], as caused by the (second) scaling of the mature separable landscape, that rules the surfaces in the upper reaches of the river valleys and in the direction perpendicular to the main rivers.

6.3 The Scaling of Exceedence Probabilities

The probability $P[A \geq a]$ that the area A drained by a channel exceeds a is known as the exceedence probability. Empirical values for this probability when it is approximated by the power law

$$P[A \geq a] \approx a^{\rho}, \tag{39}$$

which is a good approximation to observed variation, are close to -0.4 (see, for example, [28]).

Using the scaling relations in the directed percolation (DP) perpendicular to the meanderings of the main river and the two scaling above we can easily deduce this scaling of the exceedence probability once we know Hack's law. The idea to keep in mind is that a river basin is similar to a pure DP network, in that when two branches of the river network approach each other, a small subbasin of the larger river basin is created. The subbasins without a river branch are the voids of the DP network, discussed above. The time at which two branches originating at the same location join, which is called the self-interaction time β_{si}, can be measured in terms of the length l of the main river branch. We let the dimension of the network along the main river branch (backbone of the DP) be one and the dimension in the cut perpendicular to the main river branch be χ as above. Then

$$P[A \geq a] = P[l \geq t]$$

since by Hack's Law $A = l^{1+\chi}$ and $a = t^{1+\chi}$, t being the length of the main branch of the subbasin having area a. However

$$\int_0^l P[l \geq t]dt = \int_0^l t^{-\beta_{si}} dt \sim l^{1-\beta_{si}},$$

and

$$1 - \beta_{si} = -\chi,$$

see [33].[22] Thus using Hack's law again we obtain

$$P[A \geq a] \approx l^{-\chi} \tag{40}$$

$$\approx a^{-\frac{\chi}{\chi+1}} \tag{41}$$

$$\approx a^{-0.4}, \tag{42}$$

for $\chi = 0.7$. Notice that the exponent is the negative of the dynamic roughness exponent $\beta = 0.4$ in Section 5.3.

6.4 A Bridge Between Deterministic and Stochastic Modeling

Our analysis of the origin of the fundamental scaling with exponent 0.5 in the width function indicates that it arises as a result of white noise being generated by shocks in the water flow. In particular, we found that the evolution of the width functions of both the slopes and the water depths is well-characterized in terms of random walk processes. Hence the behavior and configuration of the emerging patterns of channels and valleys described by the nonlinear equations (9) and (10) and observed in the studies of Smith, Merchant, and Birnir [61] is that resulting from a random walk.

We note that these results represent an important bridge between the deterministic and stochastic lines of modeling that we discussed in Section I. ¿From the stochastic modeling literature, it is well-known that random walk models of erosion in the plane lead to relatively realistic models of key aspects of river networks (see, for example, [51, 27]) and are capable of generating branching networks that obey Horton's laws (see [28] for a summary). We believe, in fact, that the mechanisms that we have shown to underly the fundamental 0.5 scaling provide the first demonstration that physically-based PDE models lead to tree-like drainage structures whose configuration is determined by a random walk process. Hence we may view our results as providing a physically-based justification for the assumptions underlying the various stochastic modeling approaches. We note, however, that further investigation is required in order to lay a fully-acceptable foundation for such assumptions.

6.5 A Model of the Stages of Drainage Basin Evolution

Based on the preceding results and the findings of our two previous papers [58, 61], we now suggest an end-to-end theory of the evolution of landscapes that are described by our

[22] The reader should be warned the dimension parallel to the backbone $D_{||} = 0$, see [33], in this case.

advective models. In particular, this theory characterizes landscape development in terms of an evolutionary sequence involving four stages and associated time scales [23]. The first stage occurs over a relatively short time scale and is characterized by an initial, exponential increase away from the unstable initial surface (see [58]) and then a saturation by the nonlinearities. In a second stage, occurring over an intermediate time-scale, the mechanisms underlying the first scaling of 0.5 become effective and a channelized network that may be interpreted in terms of random walks emerges. Hence in this stage, the foundation of random channel network configurations are laid down. These two stages may be viewed as constituting the *youthful* stage of W.M.Davis' grand model of landscape evolution [66].

A third stage, occurring at a longer time scale, involves a self-similar cascade towards decaying separable solutions. We may view the separable solutions as a *transient attractor* (see, for example, [58]) towards which the system is drawn. The mechanisms underlying the second scaling become effective during this stage, leading to a scaling of the width function of 0.7 as well as to various scaling laws, such as Hack's Law and the Law of Exceedence Probabilities. We may view this as constituting the second, or *mature*, stage of the Davisian model. A fourth and final stage involves a short time scale over which the separable solutions collapse as the potential energy in the system approaches small values and the stable *global attractor* $H = 0$ exerts its influence. This stage may be viewed as the final, or *old age*, stage in Davis model.

We realize that may elements are missing from this model, including such processes as tectonic and isostatic uplift, gravity-driven diffusion, weathering, subsurface water movement, vegetation effects, and a host of lesser processes. Nevertheless, we believe that our advective model captures the essence of processes that lead to the evolution of dendritic, fluvial structures and their most important scaling properties.

7 Summary

The simple advective model of fluvial erosion (9)-(10) provides us with a compelling explanation for the fundamental processes present in landscape evolution. This model is capable of representing the process of channel initiation. Instabilities in unstructured initial surfaces lead to a channelization process which we are justified in viewing as the Edward-Wilkinson transient of Brownian motion. This is a noisy process: small, initial, random perturbations amplified by the nonlinear instabilities lead to noise. More importantly, however, the model (9)-(10) incorporates mechanisms capable of generating large noise. The relatively thin film of water flowing over the ridges between channels "folds over" to forms shock

[23] Clearly, the four stages do not have well-defined beginnings and endings, but they do mark significant differences in the qualitative dynamics of the system.

waves as it moves over initially convex portions of the interfluvial ridges. The derivatives of these shock waves constitute a white noise process that drives the diffusion of sediment in equation (10). The solution of this equation, linearized about the initial surface, consists of Brownian motion. Thus the projections of the channels onto the plane execute a random walk and the locations on the surface where the slope equals the average slope do a random walk on top of the first one. This explains why river networks are observed to possess the characteristics of random networks and, in particular, are characterized by the relation (55) between the bifurcation and length ratios in Hortons' laws.

This random walk is, however, not the only process influencing the shape of river basins. As the channels merge into rivers a mature landscape consisting of a pattern of concave valleys and cusp-shaped ridges emerges. These surfaces, represented by the separable solutions of the evolution equations, are characterized by another stochastic process that differs from a random walk. The reason for this difference is that the mechanism generating the noise is different from the mechanism generating noise in a channelizing surface. First, the separable surfaces are characterized by concavities whereas the ridges between the channels initially involve upper convexities. This means that shock waves, origination from surface singularities propagate down the surface. Most importantly, the vanishing of the water depth, for example on the ridge tops, on the separable surface, quenches the noise. Thus the sediment flow over the separable surfaces may be characterized as diffusion driven by quenched noise. The properties of this process, associated with directed percolation driven by quenched noise [33], are very different from those of Brownian motion.

Combining these two processes, we obtain a strikingly simple model of drainage basin evolution that is analogous to directed percolation networks. The channels execute a random walk of length l in the longitudinal direction. In a direction perpendicular to this random walk, however, the quenched directed percolation process is dominant in regions in which separable solutions have emerged, giving us the width l^{χ} of the basin, where $\chi = 0.7$ is the spatial roughness coefficient of the directed percolation process in Section 5.3. This accounts for the oval shape of river basins (see [28]), and leads to a derivation of Hack's law (see Section 6.2) as well as the observed distribution of the exceedence probabilities, see Section 6.3.

These investigations lead to interesting sets of geomorphological and mathematical problems that we intend to address in future publications. While our models capture two fundamental processes, they also raise the question as the role and significance of processes of landscape evolution that are not as yet included in our models. In particular, it is of interest to conjecture whether multifractal processes characterize more complex models. We also note that the analysis of the water flow in Section 2.4 and Appendix B provides complete information only for water flows of very small depth. If the rainfall rate, and hence the water depth, is significantly greater then more complex fluid equations are probably justi-

fied, and this may change the noise-generating mechanism and the stochastic driving of the erosion process.

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A The Generation of Shocks in the Water Flow Equation

Equation (9) describes the water flow down a water surface represented by H. We rewrite this equation in the form

$$\eta^2 \frac{\partial h}{\partial t} \; - \; \nabla \cdot \left[\mathbf{u} h^{\frac{5}{3}} \, | \, \nabla H \, |^{\frac{1}{2}} \right] \; = \; R, \tag{43}$$

where $\mathbf{u} = \frac{\nabla H}{|\nabla H|}$ denotes the unit vector in the direction of the gradient of the water surface. If we expand the gradient term, then this equation can be written in the form

$$\frac{\partial h}{\partial t} \; - \; \frac{5}{3\eta^2} h^{2/3} \, | \, \nabla H \, |^{\frac{1}{2}} \mathbf{u} \cdot \nabla h \; = \; (\nabla \cdot \mathbf{u}) | \, \nabla H \, |^{\frac{1}{2}} \frac{h^{5/3}}{\eta^2} \; + \; \frac{R}{\eta^2}, \tag{44}$$

and using the fact that

$$\nabla \cdot \mathbf{u} \; = \; \frac{\Delta H - 1}{| \, \nabla H \, |}$$

and that $\mathbf{u} \cdot \nabla h \; = \; \frac{\partial h}{\partial s}$, where s parametrizes the downstream direction given by $\mathbf{u}$, we obtain a a one-dimensional PDE

$$\frac{\partial h}{\partial t} \; - \; \frac{5}{3\eta^2} h^{2/3} \, | \, \nabla H \, |^{\frac{1}{2}} \frac{\partial h}{\partial s} \; = \; \left[\frac{\Delta H - 1}{| \, \nabla H \, |^{\frac{1}{2}}} \right] \frac{h^{5/3}}{\eta^2} \; + \; \frac{R}{\eta^2}. \tag{45}$$

It is well-known that such one-dimensional equations can either develop shocks or their solutions may blow up in finite time. We now show that for a small rainfall rate R the solutions do indeed develop a shock, which is a singularity in the s derivative, in finite time, but on concave surfaces no finite-time blow-up of h can occur.

We first set $R \; = \; 0$ and write down the characteristic equations for the equation (45),

$$\begin{gathered} \frac{dt}{d\tau} = 1, \quad \frac{ds}{d\tau} = -\frac{5}{3\eta^2} h^{2/3} \, | \, \nabla H \, |^{\frac{1}{2}} \\ \frac{dh}{d\tau} = \left[\frac{\Delta H - 1}{| \, \nabla H \, |^{\frac{1}{2}}} \right] \frac{h^{5/3}}{\eta^2} = -p(s,\tau) h^{5/3}. \end{gathered} \tag{46}$$

The function p is positive

$$p(s,\tau) = \frac{-\Delta H + 1}{| \, \nabla H \, |^{\frac{1}{2}}} > 0,$$

if $\Delta H \; < \; 1$. This condition is weaker than convexity, it simply says that the water surface cannot be too concave. Now we solve the h equation in (46) and obtain

$$h(s,t) = \frac{h_o(s)}{(1 + \frac{2}{3} h_o^{2/3}(s) P(s,t))^{3/2}} \tag{47}$$

where

$$P(s,t) = \int_0^t p(s(\tau)+s,\tau)d\tau,$$

increases along the characteristics because $p > 0$. Thus $h(s,t)$ does not blow up but decreases along the characteristics. Next we compute the characteristics making an approximation that simplifies the computation. The equation determining the characteristics is

$$\frac{ds}{d\tau} \;=\; -\frac{5}{3\eta^2}h^{2/3}\,|\,\nabla H\,|^{\frac{1}{2}} \;=\; -\frac{5}{3g(s)\eta^2}h^{2/3}, \tag{48}$$

if H evolves on a slower time-scale than h. But this was precisely the conclusion of the discussion in Section 2.1. Thus we can separate the variables and solve the above equation

$$G(s)|_{s_o}^{s} = \int_{s_o}^{s} g(s)ds = -\frac{1}{\eta^2}\int_0^t h^{2/3}(s(\tau),\tau)\,d\tau.$$

Next we expand the solution h in (48) above in powers of $h(s_o)$,

$$h(s,t) = h(s_o) - h^{5/3}(s_o)\int_0^t p(s(\tau),\tau)\,d\tau + O(h^{7/3}(s_o)),$$

and substitute the result into the equation for G, to get

$$G(s) - G(s_o) = -\frac{5}{3}\frac{h^{2/3}(s_o)}{\eta^2}t + O(h^{5/3}(s_o)/\eta^2).$$

This implies that two characteristics emanating from two distinct points $s_1 > s_o$ meet at the time

$$t = \frac{3}{5}\frac{\eta^2(G(s_1)-G(s_o))}{h^{2/3}(s_1)-h^{2/3}(s_o)} + O(h(s_o)). \tag{49}$$

Moreover, these characteristics carry different values of h by equation (47) producing a shock at the location

$$s = G^{-1}(G(s_o) - \frac{5}{3}\frac{h^{2/3}(s_o)}{\eta^2}t + O(h^{5/3}(s_o)/\eta^2))$$

at the time t. In one-dimensional models the water depth is an increasing function of s so t is positive because G is also increasing, see the example below.

The characteristics that we obtain are not straight lines as for the nonlinear wave equation, due to the term $|\nabla H|^{1/2}$. For example when $g(s) = |\nabla H|^{-1/2} = s^{1/4}$, $G(s) = \frac{4}{5}s^{5/4}$, the characteristics are given by the formula

$$s^{5/4} \;=\; s_o^{5/4} - \frac{25}{12}\frac{h^{2/3}(s_o)}{\eta^2}t + O(h^{5/3}(s_o)/\eta^2)$$

and the shock develops at the time

$$t = \frac{12}{25} \frac{\eta^2 (s_1^{5/4} - s_o^{5/4})}{h^{2/3}(s_1) - h^{2/3}(s_o)}.$$

The analysis is similar for small values of R, that can be treated perturbatively, but if the rainfall rate becomes large, it dominates and prevents shocks from developing as shown above. However, if the surface H has points where the derivative of ∇H has a singularity then these singularities cause shocks in h which propagate downstream. This is, however, a linear not a nonlinear phenomenon and it also occurs in the linearization of the equation (43). In fact the evolution of the shock in equation (45) for large R is similar to the shock propagation in the linear equation in Appendix B and we refer to this Appendix for the similar derivation for equation (9), for large R. In summary, the water flow equation possesses *two mechanisms producing shocks* in the water surface, one caused by the nonlinearities in h the other by the singularities in the water (or land) surface H. We may interpret these latter shocks as rapids or knick points, that cause shocks in the water depth and these shocks propagate downstream.

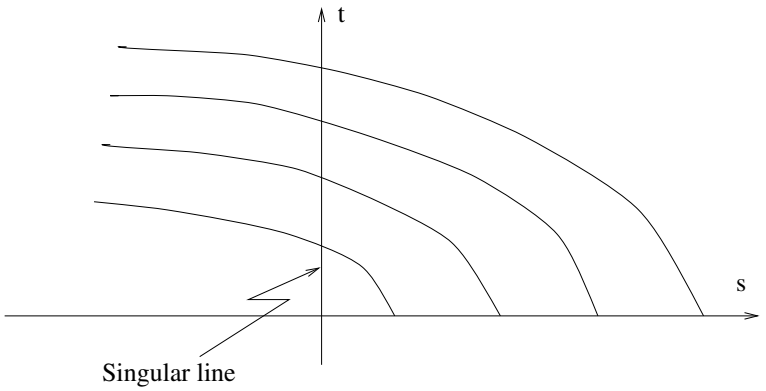

Figure 12: The characteristics for the shock caused by surface singularities.

B Surface Singularities and Shocks

We provide an illustrative example of how shocks and rarefaction waves are generated, respectively, on convex and concave separable surfaces. We note that these examples may be generalized to non-separable surfaces. We consider the one-dimensional hyperbolic equation for the evolution of the variation of the water surface

$$\eta^2 \frac{\partial v}{\partial t} \; + \; \frac{5}{3}\frac{\partial}{\partial s}(-h_1^{\frac{2}{3}}|\nabla H_1|^{1/2} v) \; = \; f(x,y,t), \tag{50}$$

where

$$\frac{1}{g(s)} = -\frac{5}{3}h_1^{\frac{2}{3}}|\nabla H_1|^{1/2} = -\eta^2 \frac{\partial q_w}{\partial h}$$

is either a strictly decreasing or a strictly increasing function. We consider approximations to initial and separable landscapes in which we assume that the landscape is locally uniform in directions perpendicular to the gradient of the water surface.

Close to a separable landscape uniform in the direction, $\mathbf{u}_w^{\perp}$, where $\mathbf{u}_w = -\frac{\nabla H}{|\nabla H|}$,[24]

$$g(s) = -s^{-1/4}. \tag{51}$$

is a reasonable model. We make the change of variables

$$a = \frac{5}{3}h_1^{\frac{2}{3}}|\nabla H_1|^{1/2} v = -\frac{v}{g(s)}$$

[24]The separable landscape is of course not uniform in the direction of $\mathbf{u}^{\perp}$ but this example still gives the right information.

then the equation can be written in the form

$$\frac{\partial a}{\partial t} + \frac{1}{g(s)\eta^2}\frac{\partial a}{\partial s} = f(x,y)/\eta^2. \tag{52}$$

If we now let

$$\frac{ds}{d\tau} = \frac{1}{g(s)\eta^2},$$

define the characteristics and $g = \frac{\partial G}{\partial s}$, then equation (52) can be solved along characteristics,

$$a(s,t) = a_0(G^{-1}(G(s) - \frac{t}{\eta^2})) + \frac{1}{\eta^2}\int_0^t f(G^{-1}(G(s) - \frac{t-\tau}{\eta^2}), \tau)d\tau.$$

The important equation is the equation for the characteristics

$$\frac{ds}{d\tau} = \frac{1}{g(s)\eta^2} = -\frac{5}{3\eta^2}h_1^{\frac{2}{3}}|\nabla H_1|^{1/2}. \tag{53}$$

It says that if the derivative of $|\nabla H|^{1/2}$ has a singularity at the origin then the derivative of h will obtain a singularity there and this singularity will propagate downstream along the characteristics

$$s = G^{-1}(G(s_o) - \frac{t}{\eta^2}),$$

see Figure 12. This constitutes the second type of shock formation, the shocks that are caused by singularities, rapids or knick points, in the surface, in distinction to the first type of shock formation discussed in Section 2.4.

C The Scaling of the Gradient

In this appendix we explain how the presence of a profile such as the initial uniform surface with constant slope, or the profile of the separable landscape, affects the scaling of the gradient of the water surface $|\nabla H|$. Consider

$$\begin{aligned} |\nabla H|(y+x) &= [H_x^2(y+x) + H_y^2(y+x)]^{1/2} \\ \sim [(ex^\chi + O(x^{2\chi}) + \cdots)^2 &+ (c + dx^\chi + O(x^{2\chi}))^2]^{1/2} \\ = (c^2 + 2cdx^\chi &+ O(x^{2\chi}))^{1/2}, \end{aligned}$$

in a statistical sense where $c, d, e,$ are constants. By the inequality,

$$(a+b)^{1/2} \leq a^{1/2} + b^{1/2},$$

$$\begin{aligned} < (|\nabla H|(y+x) - |\nabla H|(y))^2 >^{1/2} &= (c^2 + 2cdx^\chi + O(x^{2\chi}))^{1/2} - c \\ \leq (2cdx^\chi + O(x^{2\chi}))^{1/2} &= x^{\chi/2}(2cd + O(x^\chi))^{1/2}. \end{aligned}$$

Similarly

$$\begin{aligned} < (|\nabla H|(y+x) - |\nabla H|(y))^2 >^{1/2} &= (c^2 + 2cdx^\chi + O(x^{2\chi}))^{1/2} - c \\ \geq (2cdx^\chi + O(x^{2\chi}))^{1/2} &= -x^{\chi/2}(2cd + O(x^\chi))^{1/2}. \end{aligned}$$

by the inequality

$$(a+b)^{1/2} \geq a^{1/2} - b^{1/2}.$$

This shows that the statistical scaling of $|\nabla H|$ has the exponent $\chi/2$, whenever χ is the spatial roughness exponent.

D River Basins and Horton's Laws

Consider the projection of the river network onto the $x - y$ plane. Whereas the roughening of the surface above is controlled by the width function, the lateral location of a channel is determined by a pure random walk in the downridge directions. Since the flow is in the downstream direction, permissible motions at each step occur in an arc subtending 180^o. The corresponding exponents for this random walk are

$$\beta = \frac{1}{2}, \;\; \chi = 1, \;\; z = 2,$$

where β is now just the Hurst exponent. Now the tributary channels sit in the 2-dimensional plane and the fractal dimension of the river network is $D_f = 2 - \chi = 1$. The avalanche dimension is $D = 1 + \chi = 2$. If we take a one-dimensional slice perpendicular to the direction of the channel, then its position on this slice is that of a one-dimensional random walker and the water surface above this one-dimensional slice is determined by another random walker in the vertical direction (see Section 2.5.)

This analysis implies that many of the properties of river networks may be derived from a random walk. For example, the avalanche dimension for a pure random walk also gives us a simple proof of a well-known link between the Horton relations for streams. Consider the Strahler-Horton ordering for streams [25] Horton [46] discovered that if $N(a)$ denotes the number of streams of order a and $L(a)$ is their mean length then the ratios

$$\frac{L(a+1)}{L(a)} = R_L, \;\; \frac{N(a)}{N(a+1)} = R_B, \tag{54}$$

are approximately constant over the same river networks, where R_L is called the length ratio and R_B is called the bifurcation ratio. In [8], this was related to the avalanche (embedding) dimension of a river network

$$D = \frac{ln(R_B)}{ln(R_L)} \; = \; 2,$$

by the value of D computed above. Thus we obtain

$$R_B = R_L^2. \tag{55}$$

This relationship is born out by observations where typically $R_B \sim 4$ and $R_L \sim 2$.

[25] 1. streams without tributaries have order one; 2. when two streams of order a join, they form a stream of order $a + 1$; 3. when two stream of different order join, the resulting stream inherits the higher order of the two.

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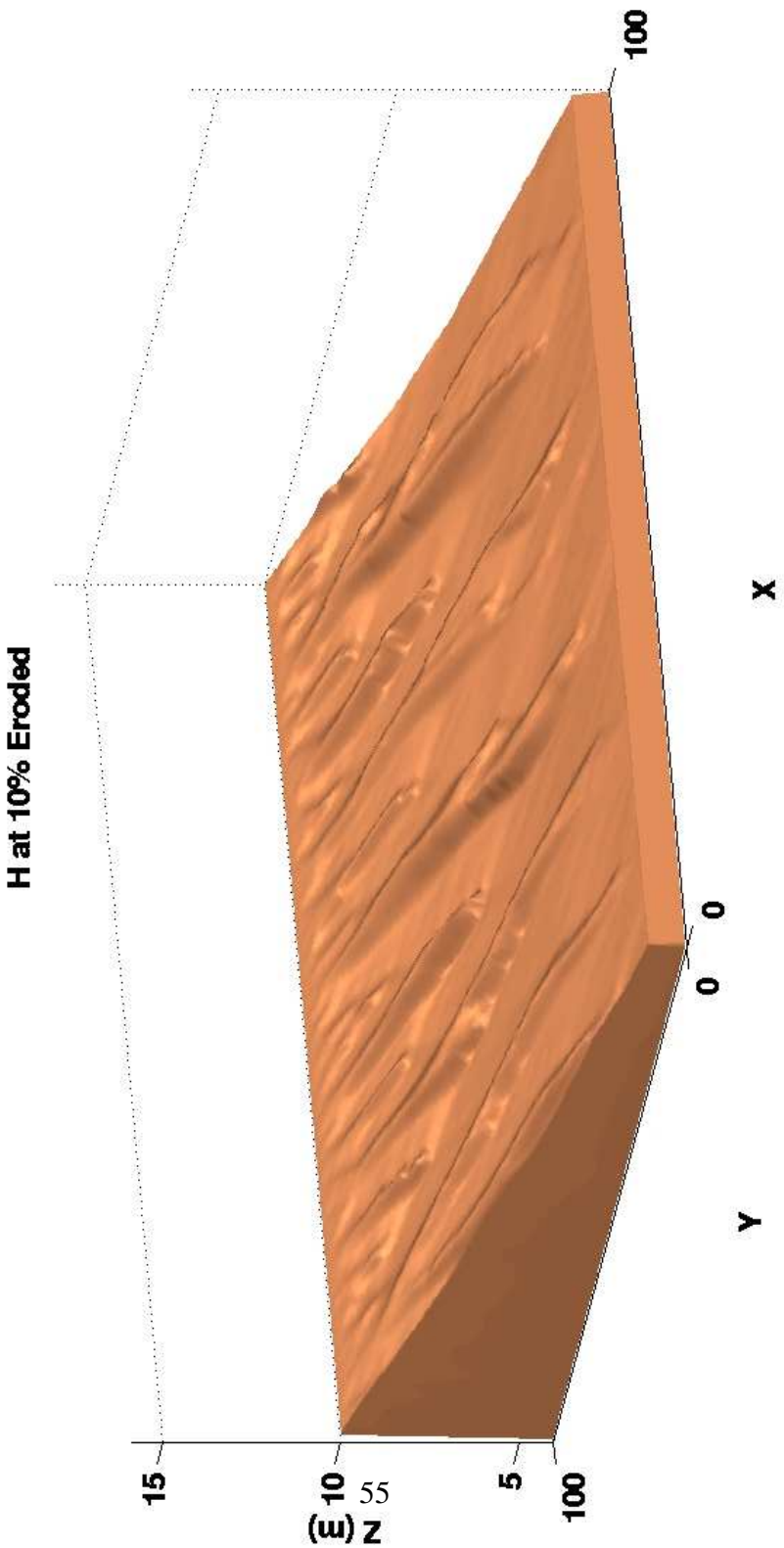

Figure 1: The Water surface H of a Typical Run at 10% of the Surface Eroded

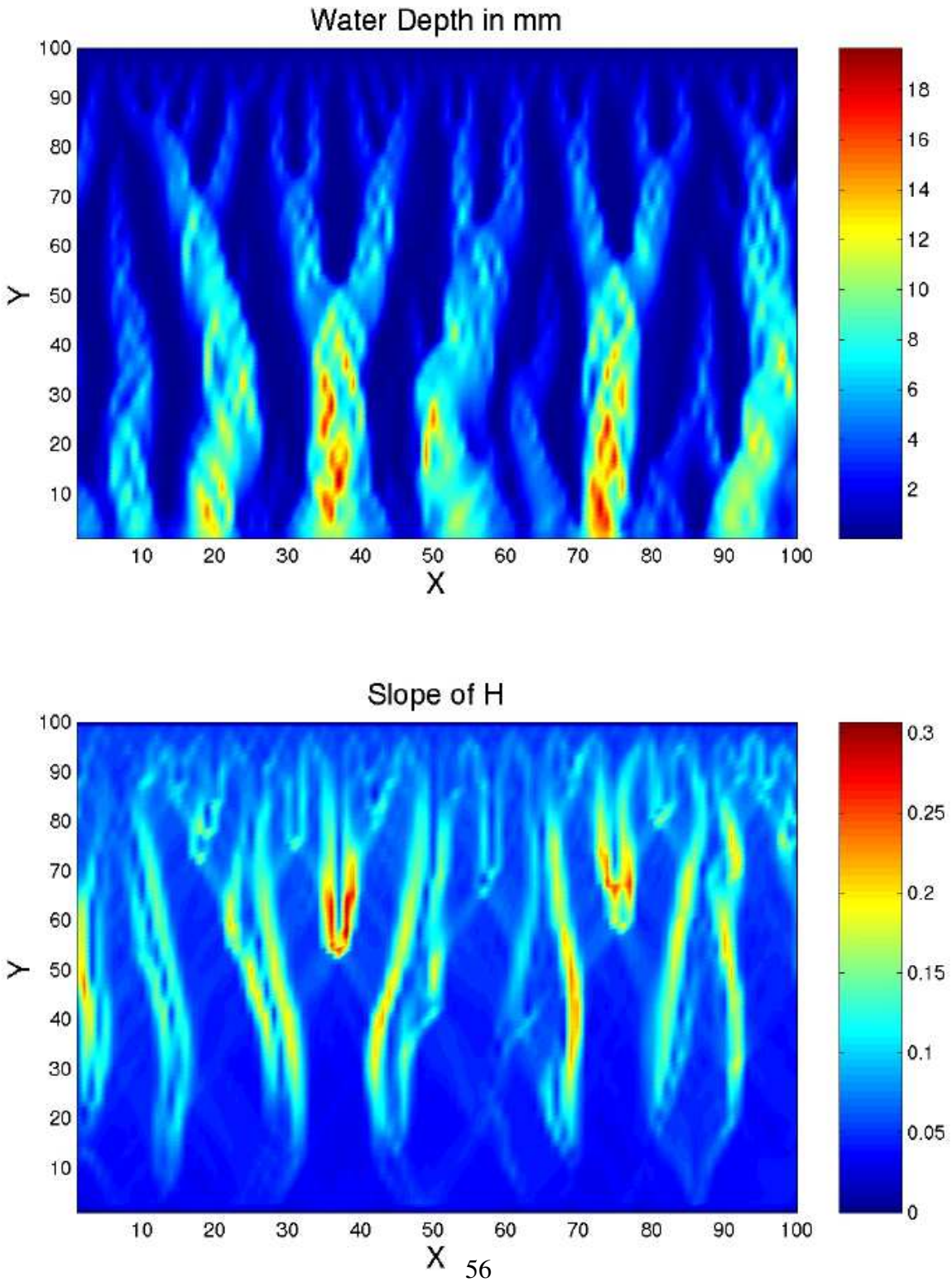

Figure 2: The Water Depth h and the Slope of the Water Surface $\mid \nabla H \mid$ of a Typical Run at 10% of the Surface Eroded

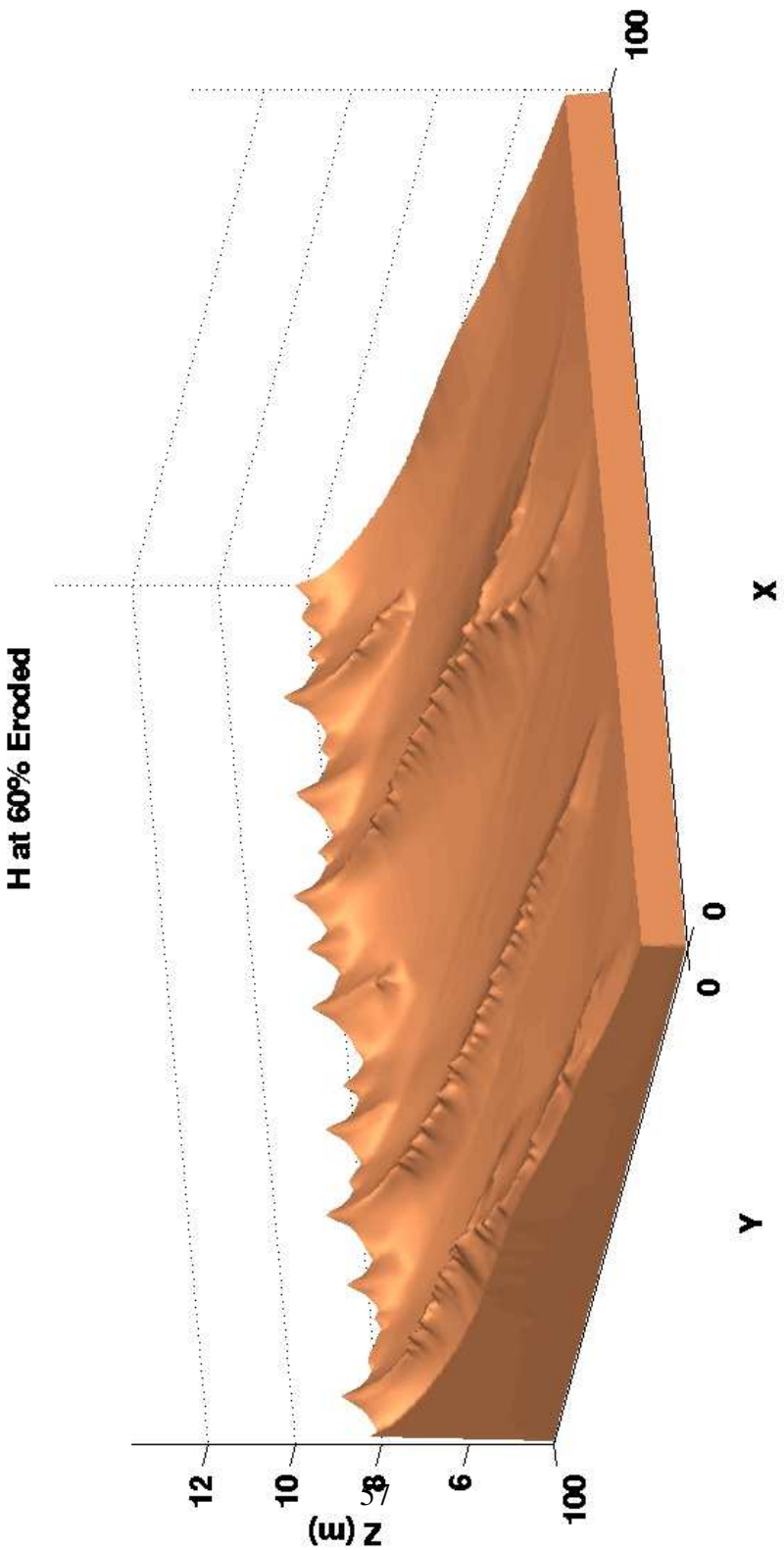

Figure 3: The Water surface H of a Typical Run at 60% of the Surface Eroded

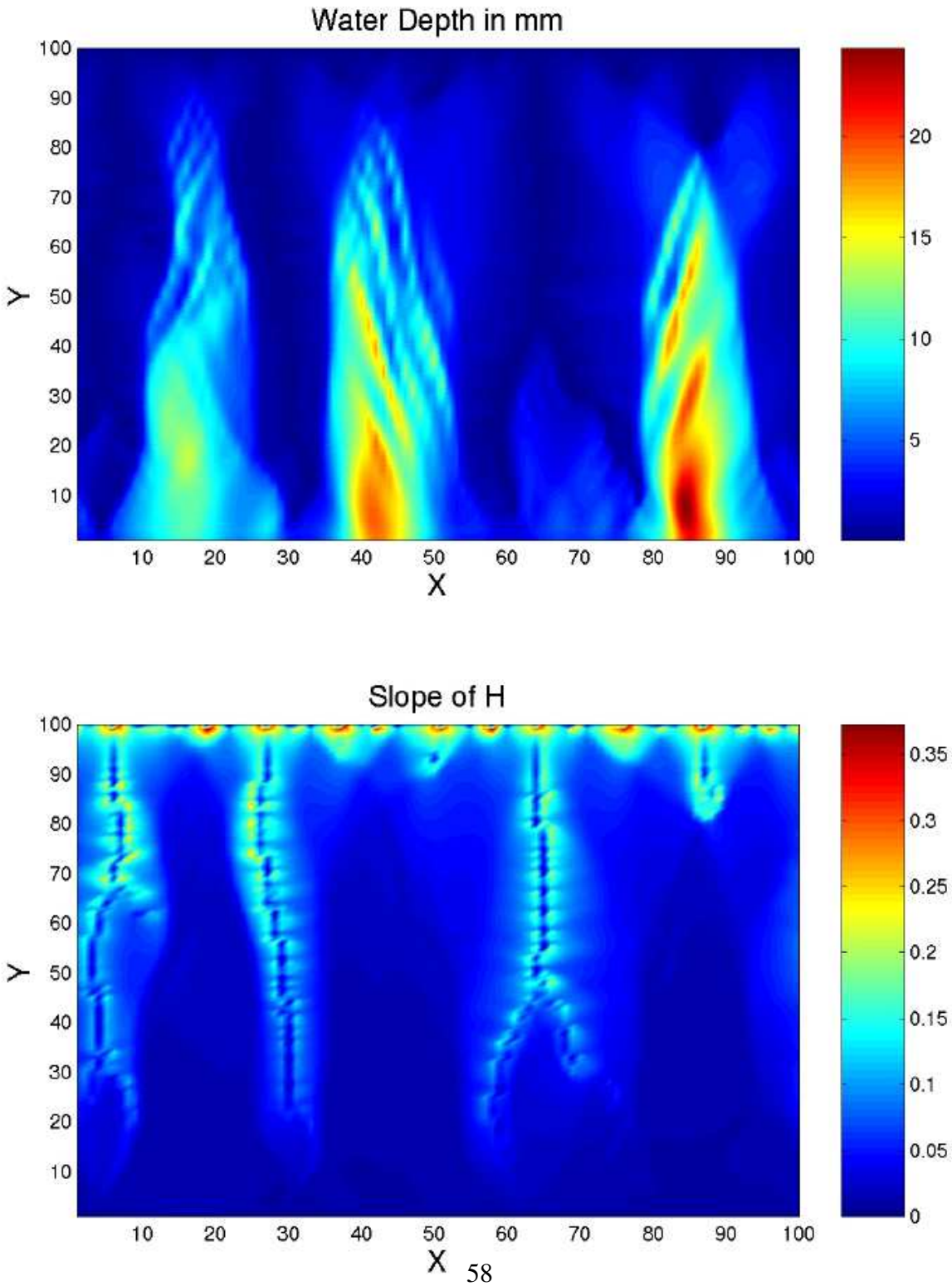

Figure 4: The Water Depth h and the Slope of the Water Surface $\mid \nabla H \mid$ of a Typical Run at 60% of the Surface Eroded