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The Spectrum of a Transfer Matrix for Loops

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Abstract. We study the spectral properties of the transfer matrix for a gonihedric random surface model on a three-dimensional lattice. The transfer matrix is indexed by generalized loops in a natural fashion and is invariant under a group of motions in loop-space. The eigenvalues of the transfer matrix can be evaluated exactly in terms of the partition function, the internal energy and the correlation functions of the two-dimensional Ising model and the corresponding eigenfunctions are explicit functions on loop-space.

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1 Introduction

The triviality of simple random surface models on the lattice, due to the dominance of branched polymers [1], and the non-scaling of the string tension in analogous Euclidean invariant models based on dynamical triangulations [2] made it imperative to study models with an action depending on the extrinsic curvature of surfaces. For a review of work in this area up to 1997, see [3].

One of the models with extrinsic curvature dependent action is the so-called gonihedric random surface model introduced in [4] and further studied in [5, 6, 7]. This model is unstable in the simplest cases [8]. However, there is a lattice discretization of the model [9, 10] which is not plagued by stability problems and is more amenable to analytical study, see [11]-[16]. Another lattice model with similar properties was introduced and studied in [17].

In [18] a simplified version of the three-dimensional gonihedric lattice model was studied and the two largest eigenvalues of the transfer matrix were determined. In this paper we finish the calculation of the spectrum of the transfer matrix for the model and find all the eigenfunctions explicitly.

2 Transfer matrix for loops

Let T_3 denote a sublattice of $\mathbf{Z}^3$ of size $M \times M \times N$ with periodic boundary conditions. We shall think of the third coordinate direction where the lattice has extension N as the 'time' direction. The configuration space of the system we wish to study is a collection of subsets of the plaquettes in T_3, denoted $\mathcal{M}$, which we refer to as singular surfaces or simply as *surfaces*. A collection of plaquettes M belongs to $\mathcal{M}$ if and only if any link in a plaquette in M belongs to an even number of plaquettes in M. This means that the surfaces in $\mathcal{M}$ do not overlap themselves but they can intersect themselves at right angles and they are closed. Note that the surfaces need not be connected. It is not hard to see that the conguration space $\mathcal{M}$ is identical to the configuration space of the three-dimensional Ising model on the periodic lattice T_3. The surfaces are in one to one correspondence with phase boundaries.

We say that a link ℓ in a surface $M \in \mathcal{M}$ is an *edge* if ℓ is contained in exactly two plaquettes that meet at a right angle. We denote the collection of all edges in

M by $\mathcal{E}(M)$. The action $S(M)$ of the surface M is defined as the total number of edges in M

$$S(M) = \#\mathcal{E}(M). \tag{1}$$

Note that no action is associated with those links where the surfaces crosses itself. The partition function is given by

$$Z(\beta) = \sum_{M \in \mathcal{M}} e^{-\beta S(M)}. \tag{2}$$

We define an *intermediate plane* in $\mathbf{R}^3$ to be a plane that lies parallel to but in between planes where one of the coordinates takes an integer value. If we slice a surface M (regarded as a subset of $\mathbf{R}^3$) by an intermediate plane we obtain a collection of links which we shall call generalized loops or simply *loops*. The collection of all possible loops that can arise in this way will be denoted Π. A collection of links P in an intermediate plane belongs to Π exactly when each vertex in P belongs to an even number of links in P. We note that the loops Π coincide with the phase boundaries that arise in the two-dimensional Ising model. It will not come as a surprise that this close connection to the Ising model will make the present model exactly soluble in a very strong sense.

For $P \in \Pi$ we let $|P|$ denote the number of links in P. Note that if $P_1, P_2 \in \Pi$, then the symmetric difference

$$(P_1 \cup P_2) \setminus (P_1 \cap P_2) \equiv P_1 \triangle P_2 \in \Pi \tag{3}$$

is again a loop in Π. It is easy to check that

$$|P_1 \triangle P_2| = |P_1| + |P_2| - 2|P_1 \cap P_2| \tag{4}$$

and $d(P_1, P_2) = |P_1 \triangle P_2|$ is a metric on Π.

Let $k(P)$ denote the number of corners of P, i.e. the number of vertices of order 2 whose adjacent links are at right angles. Let us denote by $\mathcal{H}$ the space of real-valued functions on Π. Let K_β be the operator on $\mathcal{H}$ whose matrix (kernel) is given by [11]

$$K_\beta(P_1, P_2) = \exp\left(-\beta(k(P_1) + k(P_2) + 2|P_1 \triangle P_2|)\right). \tag{5}$$

With this notation the partition function of the model can be expressed as

$$Z_N(\beta) = \mathrm{Tr}\ K_\beta^N \tag{6}$$

by slicing surfaces in intermediate planes orthogonal to the time direction. This partition function can be evaluated if we ignore the interesection term $|P_1 \cap P_2|$ in the exponent of K_β or if we replace $k(P_1) + k(P_2)$ by $k(P_1 \triangle P_2)$ [11]. In this approximation the model becomes a stack of noninteracting two-dimensional models. The critical beahaviour of the model is identical to that of the two-dimensional Ising model and the free energy can be computed exactly by standard methods [19].

Here we will make an approximation different from the one of [11]. We drop the curvature terms in the action and study the simplified transfer matrix

$$K_\beta(P_1, P_2) = \exp\left(-2\beta(|P_1 \triangle P_2| - M^2)\right), \tag{7}$$

where we have inserted a normalization factor M^2 in the exponent for later convenience. Dropping the curvature term is the same as neglecting the action associated with edges in the time-direction. We note the mathematical analogy of the transfer matrix (7) with the transition function $\exp(-\beta|x-y|)$, $x, y \in \mathbf{R}^d$, for random walk in $\mathbf{R}^d$. This makes it reasonable to regard K_β as describing the diffusion of loops.

With abuse of notation we denote the new transfer matrix (7) by the same symbol as the old one (5). This should not cause any confusion since we stick to the notation of Eq. (7) in the sequel.

3 Eigenvalues of the transfer matrix

In this section we solve the eigenvalue problem

$$\sum_{Q \in \Pi} K_\beta(P, Q)\psi_i(Q) = \Lambda_i(\beta)\psi_i(P), \tag{8}$$

$\psi_i \in \mathcal{H}$. We prove that only the eigenvalues depend on β, not the eigenfunctions. The eigenvalues can all be expressed in terms of the partition function, the internal energy and the spin correlation functions of the two-dimensional Ising model. The eigenfunctions are explicit functions on Π which can be normalized to take only the values 1 and -1.

Note first that

$$\begin{aligned} \sum_Q K_\beta(P, Q) &= \sum_Q e^{-2\beta(|Q| - M^2)} \\ &= \Lambda_0(\beta), \end{aligned} \tag{9}$$

where we have in the first step shifted the summation variable from Q to $P \triangle Q$ (permissible since the mapping $P \mapsto P \triangle Q$ is bijective) and $\Lambda_0(\beta)$ is the partition function of a two-dimensional Ising model on a periodic $M \times M$ lattice. This lattice will be denoted T_2. The loops Q are the phase boundaries of the Ising model and the Ising spin variables sit on the lattice dual to T_2.

It follows from Eq. (9) that the constant function is an eigenfunction of K_β and $\Lambda_0(\beta)$ is the corresponding eigenvalue. Since all matrix elements of K_β are positive and all entries in the eigenvector corresponding to Λ_0 have the same sign, it follows from the Perron–Frobenius theorem that Λ_0 is simple and it is the largest eigenvalue of K_β. We conclude [18] that the free energy per site in the present model is the same as the free energy per site in the two-dimensional Ising model. In particular, the two models have the same critical point and the same specific heat. However, the correlations are different, and we proceed to the calculation of the next eigenvalue of K_β.

Let us introduce the notation

$$\langle P|Q\rangle \equiv 2M^2 - 2|P \triangle Q|, \tag{10}$$

so $K_\beta(P,Q) = \exp(\beta\langle P|Q\rangle)$. We find it convenient to think of $\langle P|Q\rangle$ as the 'inner product' between the loops P and Q even though the set of loops is of course not a linear space. We shall refer to $\langle P|Q\rangle$ as the *invariant product* on Π.

If P is a loop we let $\bar{P}$ denote its complement, i.e. the loop made up of exactly those links in T_2 that P does not contain. We let 0 denote the empty loop and U the loop that contains all links in the lattice. Then $\bar{0} = U$ and $\bar{P} = P \triangle U$. The invariant product is clearly symmetric in its two arguments and it has the following properties:

$$\langle P|P\rangle = 2M^2 \tag{11}$$

$$\langle P|0\rangle = 2M^2 - 2|P| \tag{12}$$

$$\langle P|\bar{Q}\rangle = -\langle P|Q\rangle. \tag{13}$$

One can easily construct a finite dimensional inner product space which contains all loops in a natural fashion such that the inner product of loops coincides with the invariant product and the loops lie on a sphere of radius $\sqrt{2}M$.

The invariant product is invariant under a nonabelian group $\mathcal{G}$ of motions in loop-space generated by

- the translations of loops
$$T_a : P \mapsto P + a, \tag{14}$$
$a \in T_2$,
- 'generalized antipodal maps'
$$A_Q : P \mapsto P \triangle Q, \tag{15}$$
$Q \in \Pi$,
- reflections and rotations in T_2.

Invariance means that $\langle gP|gQ\rangle = \langle P|Q\rangle$ for any $g \in \mathcal{G}$. As explained above we can view the set of loops as a sphere and we are free to regard the empty loop as the north-pole and the complete loop U at the south-pole. Loops increase in size as one moves from north to south. The mapping $P \mapsto \bar{P}$ is in this picture the usual antipodal map and the mappings A_Q all have the property $A_Q^2 = I$ where I is the identity map.

The group $\mathcal{G}$ acts on the functions in $\mathcal{H}$ in a natural way:

$$g\psi(P) = \psi(g^{-1}P). \tag{16}$$

Let O_n denote the linear operator on $\mathcal{H}$ with matrix elements

$$O_n(P, Q) = \langle P|Q\rangle^n. \tag{17}$$

Then O_n is invariant under the action of $\mathcal{G}$ on $\mathcal{H}$, i.e.

$$gO_n g^{-1} = O_n \tag{18}$$

and the operators O_n commute with each other. Since they are symmetric it follows that they have common eigenvectors and the same applies of course to the transfer matrix

$$K_\beta = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} O_n. \tag{19}$$

This proves that the eigenvectors of K_β are independent of β.

In view of the invariance of the transfer matrix under $\mathcal{G}$ it is natural to look for eigenvectors which depend on P only via the invariant product. Let us define

$$\psi_Q(P) = \langle P|Q\rangle. \tag{20}$$

We claim that

$$K_\beta \psi_Q = \Lambda_1(\beta)\psi_Q \tag{21}$$

for any $Q \in \Pi$, where

$$\Lambda_1(\beta) = \frac{1}{2M^2}\sum_P e^{\beta\langle P|0\rangle}\langle P|0\rangle, \tag{22}$$

so the eigenvalue ratio $\Lambda_1\Lambda_0^{-1}$ is minus the internal energy per link of the Ising model on T_2 [18].

In order to prove Eqs. (21) and (22) it is convenient to introduce a function η_ℓ on Π, where ℓ is a link in the lattice T_2, defined as

$$\eta_\ell(P) = \begin{cases} -1 & \text{if } \; l \in P, \\ 1 & \text{if } \; l \notin P. \end{cases} \tag{23}$$

Note that

$$\langle P|Q\rangle = \sum_\ell \eta_\ell(P\triangle Q), \tag{24}$$

where the sum runs over all links in the lattice T_2. Consider the function

$$\Phi_{Q,l} : P \mapsto \eta_\ell(P\triangle Q) \tag{25}$$

on Π. If $\ell \notin Q$ then $\ell \in P\triangle Q$ if and only if $\ell \in P$. Similarly, if $\ell \in Q$ then $\ell \in P\triangle Q$ if and only if $\ell \notin P$. We conclude that

$$\eta_\ell(P\triangle Q) = \eta_\ell(P)\eta_\ell(Q). \tag{26}$$

It follows that

$$\begin{aligned} \sum_P e^{\beta\langle 0|P\rangle}\eta_\ell(P\triangle Q) &= \sum_P e^{\beta\langle 0|P\rangle}\eta_\ell(P)\,\eta_\ell(Q) \\ &= \frac{1}{2M^2}\sum_P e^{\beta\langle 0|P\rangle}\langle 0|P\rangle\,\eta_\ell(Q). \end{aligned} \tag{27}$$

The last equality is obtained by observing that the sum

$$\sum_P e^{\beta\langle 0|P\rangle}\eta_\ell(P) \tag{28}$$

is independent of ℓ due to the translational and rotational invariance of the invariant product and using Eq. (24). This proves Eqs. (21) and (22) since it suffices to verify

$$\sum_{P} K_{\beta}(P', P)\langle P|Q\rangle = \Lambda_1 \langle P'|Q\rangle \tag{29}$$

for $P' = 0$ due to the invariance of K_{β} under $\mathcal{G}$. In fact we have shown that the functions $\Phi_{Q,\ell}$ are all eigenfunctions of K_{β} with the eigenvalue Λ_1.

We now use the elementary functions $\Phi_{Q,\ell}$ to construct the higher eigenfunctions of K_{β}. Let ℓ_1 and ℓ_2 be two distinct links. Define

$$\Phi_{Q_1 Q_2,\ell_1 \ell_2}(P) = \Phi_{Q_1,\ell_1}(P)\Phi_{Q_2,\ell_2}(P). \tag{30}$$

Then, by Eq. (26),

$$\left(K_{\beta}\Phi_{Q_1 Q_2,\ell_1 \ell_2}\right)(0) = \Lambda_2(\ell_1,\ell_2)\Phi_{Q_1 Q_2,\ell_1 \ell_2}(0), \tag{31}$$

i.e. $\Phi_{Q_1 Q_2,\ell_1 \ell_2}$ is an eigenfunction of K_{β} with eigenvalue

$$\Lambda_2(\ell_1,\ell_2) = \sum_{P} e^{\beta\langle 0|P\rangle}\eta_{\ell_1}(P)\eta_{\ell_2}(P). \tag{32}$$

Note that $\Lambda_2(\ell_1,\ell_2)$ depends on the links ℓ_1 and ℓ_2 but only on their relative orientation and the distance between them. Of course the eigenvalue also depends on β but we suppress this from our notation.

Similarly, if the links $\ell_1, \ell_2, \ldots, \ell_n$ are all distinct, then

$$\Phi_{Q_1\ldots Q_n,\ell_1\ldots\ell_n}(P) = \prod_{i=1}^{n}\Phi_{Q_i,\ell_i}(P) \tag{33}$$

is an eigenfunction of K_{β} with the eigenvalue

$$\Lambda_n(\ell_1,\ldots,\ell_n) = \sum_{P} e^{\beta\langle 0|P\rangle}\prod_{i=1}^{n}\eta_{\ell_i}(P). \tag{34}$$

The eigenvalue is symmetric under permutations of the ℓ_i's and simultaneous lattice rotations or translations of the links.

If ℓ is a link in the lattice T_2 let σ_{ℓ}^1 and σ_{ℓ}^2 be the two Ising spin variables on the dual sites adjacent to the link ℓ. Then $\eta_{\ell}(P) = \sigma_{\ell}^1(P)\sigma_{\ell}^2(P)$ where $\sigma_{\ell}^i(P)$ is the value taken by σ_{ℓ}^i in the spin configuration corresponding to the loop P. The expectation value for an Ising model on T_2 is given by

$$\langle(\,\cdot\,)\rangle = \Lambda_0^{-1}\sum_{P} e^{\beta\langle P|0\rangle}(\,\cdot\,) \tag{35}$$

so

$$\Lambda_n(\ell_1,\ldots,\ell_n) = \Lambda_0\,\langle\prod_{i=1}^{n}\sigma^1_{\ell_i}\sigma^2_{\ell_i}\rangle. \tag{36}$$

The correlation inequalities

$$\langle\prod_{i=1}^{n}\sigma^1_{\ell_i}\sigma^2_{\ell_i}\rangle \geq \langle\prod_{i=1}^{k}\sigma^1_{\ell_i}\sigma^2_{\ell_i}\rangle\langle\prod_{i=k+1}^{n}\sigma^1_{\ell_i}\sigma^2_{\ell_i}\rangle \tag{37}$$

now imply

$$\Lambda_n(\ell_1,\ldots,\ell_n)\Lambda_0 \geq \Lambda_k(\ell_1,\ldots,\ell_k)\Lambda_{n-k}(\ell_{k+1},\ldots,\ell_n). \tag{38}$$

We expect that

$$\Lambda_n(\ell_1,\ldots,\ell_n) < \Lambda_1, \tag{39}$$

for any $n > 1$ but do not have a general proof of this inequality. It can be checked in special cases using the explicit form of the two spin correlation function [20] and the cluster property of correlations.

4 Multiplicities

As remarked above, the largest eigenvalue Λ_0 is simple by the Perron–Frobenius theorem. Let us consider the first nontrivial eigenvalue Λ_1 with eigenfunctions $\Phi_{Q,\ell}$. Since $\Phi_{Q,\ell}$ is a multiple of $\Phi_{0,\ell}$ by the constant $\eta_\ell(Q)$ it suffices to consider the functions $\Phi_{0,\ell} = \eta_\ell$. We claim that these functions are linearly independent so the multiplicity of Λ_1 is at least $2M^2$.

Suppose there are constants c_ℓ such that

$$\sum_\ell c_\ell\,\eta_\ell(P) = 0 \tag{40}$$

for any $P \in \Pi$. We adopt the convention that sums on ℓ run over all links in T_2 unless otherwise specified. Then

$$-\sum_{\ell\in P} c_\ell + \sum_{\ell\notin P} c_\ell = 0. \tag{41}$$

By taking $P = 0$ we also obtain

$$\sum_\ell c_\ell = 0. \tag{42}$$

It follows that

$$\sum_{\ell\in P} c_\ell = 0 \tag{43}$$

for any $P \in \Pi$. Let now x and y be two different lattice points in T_2 and suppose P is made up of two simple nonintersecting curves γ_1 and γ_2 from x to y. Then clearly

$$\sum_{\ell \in \gamma_1} c_\ell = -\sum_{\ell \in \gamma_2} c_\ell. \tag{44}$$

Let γ_3 be one more simple curve from x to y which also avoids γ_1 and γ_2. Then we can join γ_3 with either γ_1 or γ_2 to make a closed curve (i.e. a loop) in Π. We conclude that

$$\sum_{\ell \in \gamma_1} c_\ell = \sum_{\ell \in \gamma_3} c_\ell = \sum_{\ell \in \gamma_2} c_\ell \tag{45}$$

which implies that

$$\sum_{\ell \in \gamma_i} c_\ell = 0 \tag{46}$$

for $i = 1, 2, 3$. Now take x and y to be nearest neighbours and γ_1 the curve that joins them by one link ℓ_1. It follows that $c_{\ell_1} = 0$ and hence $c_\ell = 0$ for all ℓ since x and y are arbitrary.

We could have obtained the above result more easily by noting that the functions η_ℓ are mutually orthogonal in the natural inner product on $\mathcal{H}$ which we denote by $(\cdot, \cdot)$. This follows from the fact that $\eta_{\ell_1}\eta_{\ell_2}$ with $\ell_1 \neq \ell_2$ is an eigenfunction of K_β with an eigenvalue $\Lambda_2(\ell_1, \ell_2) \neq \Lambda_0$ so $\eta_{\ell_1}\eta_{\ell_2}$ is orthogonal to the constant function, i.e.

$$(1, \eta_{\ell_1}\eta_{\ell_2}) = \sum_P \eta_{\ell_1}(P)\eta_{\ell_2}(P) = 0. \tag{47}$$

Hence,

$$(\eta_{\ell_1}, \eta_{\ell_2}) = \delta_{\ell_1\ell_2}\, 2^{M^2} \tag{48}$$

since the total number of loops in Π is equal to 2^{M^2}. This can also be seen by a direct calculation.

If the inequality (39) is valid then the multiplicity of Λ_1 is exactly $2M^2$. This follows from the fact that the functions $\Phi_{Q_1 \dots Q_n, \ell_1 \dots \ell_n}$ span $\mathcal{H}$. We now prove this spanning property.

Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four different links which contain the same vertex. We shall call such a collection of links a *star*. In this case

$$\eta_{\ell_1}(P)\eta_{\ell_2}(P)\eta_{\ell_3}(P)\eta_{\ell_4}(P) = 1 \tag{49}$$

for any loop P since P contains 0, 2 or 4 of the links $\ell_1, \dots, \ell_4$. If $\ell_1, \ell_2, \ell_3, \ell_4$ are a star and $n \geq 4$ it follows that

$$\Phi_{Q_1 \dots Q_n, \ell_1 \dots \ell_n} = \pm \Phi_{Q_5 \dots Q_n, \ell_5 \dots \ell_n}. \tag{50}$$

Let $\mathcal{L} = \{\ell_1, \dots, \ell_n\}$ be a collection of links in T_2 and define

$$E_{\mathcal{L}}(P) = \prod_{\ell \in \mathcal{L}} \eta_\ell(P). \tag{51}$$

Clearly $E_{\mathcal{L}} = \pm \Phi_{Q_1 \dots Q_n, \ell_1 \dots \ell_n}$ so in order to prove the spanning property it suffices to identify 2^{M^2} mutually orthogonal functions of the form $E_{\mathcal{L}}$. We adopt the convention $E_\emptyset = 1$.

If $\mathcal{L}$ and $\mathcal{L}'$ are two collections of links we say that they are *equivalent* if the symmetric difference $\mathcal{L} \triangle \mathcal{L}'$ is a star or the symmetric difference of two or more stars. This defines an equivalence relation on the set of all collections of links. If $\mathcal{L}$ is a collection of links let $[\mathcal{L}]$ denote the equivalence class of $\mathcal{L}$. It is easy to see that the total number of elements in each equivalence class is 2^{M^2} and the number of different equivalence classes is also 2^{M^2} since the total number of collections of links from T_2 is 4^{M^2}. If we choose one $\mathcal{L}_i$ from each equivalence class $[\mathcal{L}_i]$, $i = 1, \dots, 2^{M^2}$, then

$$(E_{\mathcal{L}_i}, E_{\mathcal{L}_j}) = \sum_P \prod_{\ell \in \mathcal{L}_{ij}} \eta_\ell(P), \tag{52}$$

where $\mathcal{L}_{ij}$ is a maximal subset of $\mathcal{L}_i \triangle \mathcal{L}_j$ which contains no star. Hence, for $i \neq j$,

$$(E_{\mathcal{L}_i}, E_{\mathcal{L}_j}) = (1, E_{\mathcal{L}_{ij}}) = 0 \tag{53}$$

since $\mathcal{L}_{ij}$ is nonempty for $i \neq j$. This proves that $\{E_{\mathcal{L}_i}\}$ is a family of mutually orthogonal functions.

5 Discussion

Using the results derived in the previous section it is quite easy to calculate correlation function, i.e. functions of the form

$$G_\beta^{(N)}(P, Q) = \sum_{\partial M = P \cup Q} e^{-\beta S_1(M)}, \tag{54}$$

where the loops P and Q lie in two intermediate constant time planes separated by N lattice spacings. In Eq. (54) we sum over all surfaces whose intersection with the two intermediate planes are P and Q and $S_1(M)$ is a modified action functional, counting only edges that are orthogonal to the time direction. We have

$$G_\beta^{(N)}(P,Q) = (\delta_P, K_\beta^N \delta_Q), \tag{55}$$

where δ_P and δ_Q are delta functions in $\mathcal{H}$. Up to normalization we can interpret $G_\beta^{(N)}(P,Q)$ as the probability of having the loop P at time N given that we have Q at time 0. An easy calculation gives

$$G_\beta^{(N)}(P,Q) = 2^{-M^2} \sum_{i=1}^{2^{M^2}} E_{\mathcal{L}_i}(P \triangle Q) \Lambda_{\mathcal{L}_i}^N, \tag{56}$$

where the sum runs over the orthogonal family of functions constructed in the last section and $\Lambda_{\mathcal{L}_i}$ is the eigenvalue of K_β corresponding to $E_{\mathcal{L}_i}$.

Unfortunately it is not clear how to extend the analysis of the present paper to include the curvature terms in the action of the full model. The original transfer matrix (5) is not invariant under the group $\mathcal{G}$, only translations and rotations remain symmetries. The constant function is not an eigenfunction any more. The only result that survives is the fact that the largest eigenvalue is in this case also simple by the Perron–Frobenius theorem.

However, we have managed to solve exactly a three dimensional lattice model that describes the diffusion of loops. As a surface model it is not isotropic but can be viewed as a stack of two-dimensional Ising systems with an intereaction that is sufficiently simple for us to solve the model exactly.

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