ASYMPTOTIC ESTIMATION ON CERTAIN PRODUCT RAMSEY NUMBERS

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ABSTRACT. We will derive an asymptotic upper bound for the Product Ramsey Number $PRN(m,k,r;t)$ in the case $k=1$, r and t are arbitrary and where m is considered as a large number which tends to infinity. In simpler terms, we estimate how large R must be so that any r-coloring of the t-dimensional R-box will give a unicolor t-dimensional m-box. We do this by getting an upper bound for a function we know is actually larger than this Product Ramsey Number.

1. GENERAL SETUP AND DEFINITIONS

Referring to [1, pg 28] we define $R_t(m,r)$ to be the Product Ramsey Number $PRN(m,k,r;t)$ in the case $k=1$. Here $PRN(m,k,r;t)$ is the general Product Ramsey Number $R(r,t;k_1,\ldots,k_t;m_1,\ldots,m_t)$ where $k_1=\ldots=k_t=k$ and $m_1=\ldots=m_t=m$.

The purpose of this article is to examine $R_t(m,r)$ for arbitrary values of r and t and when m is large.

For $n\in \boldsymbol{N}$ let $[n]$ denote the set $\{1,2,\ldots,n\}$. In simple terms $R_t(m,r)$ is the least R such that if

$$f:\underbrace{[R]\times\cdots\times[R]}_{t \text{ times}}\to[r]$$

is a function (coloring) then there exists m element-subsets $S_i\subseteq[R]$ such that all elements of the t-dimensional m-box $S_1\times\ldots\times S_t$ recieve the same color by f.

Clearly for any m and r we have $R_1(m,r)=(m-1)r+1$ by usual Pigeon-Hole principle. Hence we will study $R_t(m,r)$ for $t\geq 2$.

CONVENTION: Unless otherwise stated $[R]^t$ will always mean the usual cartesian product $[R]^t=[R]\times\cdots\times[R],$ (t times) but *not* the set of t-element subsets of $[R]$ as in [2].

In order to get an upper bound for $R_t(m,r)$ we will establish an upper bound for a function $P_t(m,r)$ that we know is larger than $R_t(m,r)$:

Definition 1.1. *For $m, r, t \geq 2$ let $P_t(m, r)$ be the least integer R such that every $\lceil R^t/r \rceil$ element subset of $[R]^t$ contains a t-dimensional m-box $S_1 \times \cdots \times S_t$ where each S_i has m elements.*

To get an asymptotic upper bound for $R_t(m, r)$ later on, we bear in mind the following:

- Clearly $R_t(m, r) \leq P_t(m, r)$ for all m, r and t.
- We will bound $P_t(m, r)$ for large m by induction on t.

Definition 1.2. *Define the following:*

1. *For $\alpha_1 < \ldots < \alpha_l$ where $1 \leq l \leq t$ and $\alpha_i \in \{1, \ldots, t\}$ let*
$$\pi_{\alpha_1,\ldots,\alpha_l} : [R]^t \to [R]^l$$
$$\pi_{\alpha_1,\ldots,\alpha_l}(i_1, \ldots, i_t) = (i_{\alpha_1}, \ldots i_{\alpha_l}),$$
to be the projection onto coordinates nr. $\alpha_1, \ldots, \alpha_l$.
2. *For a subset $M \subseteq [R]^t$ and $\tilde{\imath} = (i_1, \ldots, i_{t-1}) \in [R]^{t-1}$ let*
$$M_{\tilde{\imath}} = M \cap \pi^{-1}_{1,\ldots,t-1}(i_1, \ldots, i_{t-1}).$$
Hence we get a disjoint partition of M as $M = \bigcup_{\tilde{\imath} \in [R]^{t-1}} M_{\tilde{\imath}}$.

2. Necessary tools

Let $f : [R]^t \to [r]$ be a coloring function. If R is sufficiently large then by the Pigeon-Hole principle there is a subset $S \subseteq [R]^t$ such that $|S| \geq R^t/r$, recieving the same color by f. For each $\tilde{\imath} \in [R]^{t-1}$ let $s_{\tilde{\imath}} = |S_{\tilde{\imath}}|$, then there are exactly $\binom{s_{\tilde{\imath}}}{m}$ m-sets of $S_{\tilde{\imath}}$. Hence the total number of m-sets of $|S|$, each of which is mapped to a singleton under $\pi_{1,\ldots,t-1}$ is

$$\sum_{\tilde{\imath} \in [R]^{t-1}} \binom{s_{\tilde{\imath}}}{m}.$$

Call the set of these m-sets $S^{(m)}$. Note that $\sum s_{\tilde{\imath}} = |S| \geq R^t/r$.

In order to get an upper bound for $P_t(m, r)$ the idea is to introduce an extra integer variable x which we assume is less than R. Now, the total number of m-sets in $[R]$ is $\binom{R}{m}$. To ensure that at least $\lceil R^{t-1}/x \rceil$ elements of $S^{(m)}$ are mapped to the same m-set of $[R]$ under π_t, we must have

$$\sum_{\tilde{\imath} \in [R]^{t-1}} \binom{s_{\tilde{\imath}}}{m} \geq \left(\left\lceil \frac{R^{t-1}}{x} \right\rceil - 1 \right) \binom{R}{m} + 1$$

and hence it suffices that

$$\sum_{\tilde{\imath} \in [R]^{t-1}} \binom{s_{\tilde{\imath}}}{m} > \frac{R^{t-1}}{x} \binom{R}{m}. \tag{1}$$

On the other hand to ensure that the image of these $\lceil R^{t-1}/x \rceil$elements of $S^{(m)}$ in $[R]^{t-1}$ under $\pi_{1,\dots,t-1}$ contains a $t-1$ dimesional m-box we must by definition 1.1 have

$$R \geq P_{t-1}(m, x). \tag{2}$$

Hence if both (1) and (2) are satisfied then we can rest assure that there is a t-dimensional m-box in $[R]^t$ all of which recieve the same color under f.

We will use conditions (1) and (2) to get an upper bound for $P_t(m, r)$. First we have two technical lemmas. For convenience let us, for a fixed integer m, define the real function

$$\binom{x}{m}^* = \begin{cases} \frac{x(x-1)\cdots(x-m+1)}{m!} & \text{if } x \geq m-1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2.1. *For fixed m and C let $x_1, \dots, x_n$ be real positive variables subjected to the constraint $x_1 + \cdots + x_n \geq C$. Then $\sum_{i=1}^n \binom{x_i}{m}^*$ reaches its minimum value when $x_1 = \ldots = x_n$.*

PROOF: Clearly the minumal value of the sum is reached when $x_1 + \cdots + x_n = C$. Assume first that $x_i \geq m-1$ for each i. By the elementary method of Lagrange multipliers the minimal value is obtained when

$$\frac{\partial}{\partial x_1}\binom{x_1}{m} = \cdots = \frac{\partial}{\partial x_n}\binom{x_n}{m}. \tag{3}$$

Considering one variable x we see that $\binom{x}{m}$ is a polynomial of degree m with roots $0, 1, \dots, m-1$ and hence by Rolle's theorem the roots of the polymomial $P(x) = \frac{d}{dx}\binom{x}{m}$ of degree $m-1$ are located in the open intervals $]0;1[$, $]1;2[$, $\dots$, $]m-2;m-1[$, exactly one root in each interval. Therefore $P(x)$ is strictly increasing for $x \geq m-1$ and hence injective. By this we get from (3) that $x_1 = \cdots = x_n$ must hold.

Consider now the case where $x_i < m-1$ for some i. If we let

$$x'_i = \begin{cases} x_i & \text{if } x_i \geq m-1 \\ m-1 & \text{if } x_i < m-1 \end{cases}$$

then $C' = \sum_{i=1}^m x'_i > \sum_{i=1}^m x_i = C$. By the definition of $\binom{x}{m}^*$ since each $x'_i \geq m-1$ we have by previous argument that

$$n\binom{C/n}{m}^* \leq n\binom{C'/n}{m}^* \leq \sum_{i=1}^m \binom{x'_i}{m}^* = \sum_{i=1}^m \binom{x_i}{m}^*.$$

So in any case $\sum_{i=1}^m \binom{x_i}{m}^*$ reaches its minimal value when all the x_i's are the same.□

Lemma 2.2. *For positive integers $m, r \geq 1$ and a real number $\alpha \in]0;1[$, the largest real root of the polynomial equation*

$$X(X-r)\cdots(X-(m-1)r) = \alpha X(X-1)\cdots(X-m+1) \tag{4}$$

is asymptotically as $R = \frac{m^2(1-r)}{2\log(\alpha)}$ *when* m *tends to infinity. Here* log *is the usual logarithm with base number* $e \approx 2,71828$.

NOTE: For fixed $r \geq 2$ and $\alpha \in]0;1[$ we see that there is a sufficiently large m such that the largest root of (4) is strictly less than $\frac{m^2 r}{2\log(\alpha^{-1})}$.

PROOF: If we let

$$F_m(X) = \frac{X(X-r)(X-2r)\cdots(X-(m-1)r)}{X(X-1)(X-2)\cdots(X-m+1)}$$

then

$$F_m(m^2rY) = \frac{r^m(m^2Y)!(m^2rY-m)!}{(m^2rY)!(m^2Y-m)!} \underset{m\to\infty}{\longrightarrow} e^{\frac{1}{2}\left(\frac{1-r}{rY}\right)}$$

by Stirling's formula for estimating $n!$ for large n. We note that this limit value is equal to α when $Y = \frac{1-r}{2r\log\alpha}$.

Let A and B be real numbers such that $A < \frac{1-r}{2r\log\alpha} < B$. Since the map $Y \mapsto \frac{1}{2}\left(\frac{1-r}{rY}\right)$ is increasing we get $\frac{1}{2}\left(\frac{1-r}{rA}\right) < \alpha < \frac{1}{2}\left(\frac{1-r}{rB}\right)$. Hence we can find m_0 such that

$$F_m(m^2rA) < \alpha < F_m(m^2rB) \tag{5}$$

for all $m \geq m_0$. For r, α given let $m_1 \geq m_0$ be such that $\frac{m^2(1-r)}{2\log\alpha} > (m-1)r$ for all $m \geq m_1$. Consider now a fixed $m \geq m_1$.

- Since the map $Y \mapsto F_m(m^2rY)$ is continuous we get from (5) that there is a real z_m in the open interval $]A;B[$ such that $F_m(m^2rz_m) = \alpha$.
- Since $m^2rB > \frac{m^2(1-r)}{2\log\alpha} > (m-1)r$ and $\frac{d}{dX}F_m(X) > 0$ for $X > (m-1)r$ we have that $F_m(X)$ is increasing for $X \geq m^2rB$ and hence $F_m(X) - \alpha$ has no real root greater than m^2rB.

We conclude that the largest real root R_m of the equation $F_m(X) = \alpha$ is contained in the interval $[m^2rz_m; m^2rB[\;\subseteq\;]m^2rA; m^2rB[$, or in other words $\frac{R_m}{m^2r} \in]A;B[$. This holds for all $m \geq m_1$, hence

$$\lim_{m\to\infty} \frac{R_m}{m^2r} = \frac{1-r}{2r\log\alpha}.\square$$

In order for R to satisfy (1) it is sufficient by lemma 2.1 that R satisfies $R^{t-1}\binom{R/r}{m}^* > \frac{R^{t-1}}{x}\binom{R}{m}$, that is to say $\binom{R/r}{m}^* > \frac{1}{x}\binom{R}{m}$. Note that this condition does not depend on the dimension t at all.

3. Receiving the upper bound

From conditions (1) and (2) we therefore extract new and more simple conditions on R that ensures that the set S contains an m-box in $[R]^t$:

$$\binom{R/r}{m}^* > \frac{1}{x}\binom{R}{m} \tag{6}$$

$$R \geq P_{t-1}(m,x) \tag{7}$$

where x is some integer variable $< R$. Since both sides of (6) are polynimials of degree m, in order for (6) to hold for all sufficiently large R, $x > r^m$ must hold. Since now (6) can be written in the form

$$R(R-r)\cdots(R-(m-1)r) > \frac{r^m}{x}R(R-1)\cdots(R-m+1)$$

we have by lemma 2.2 that for large enough m, it suffices for R to satisfy $R \geq \frac{m^2 r}{2\log(x/r^m)}$ in order for (6) to hold.

Hence we see that if $x > r^m$ and

$$R \geq \max\left(P_{t-1}(m,x), \frac{m^2 r}{2\log(x/r^m)}\right)$$

then every $\lceil \frac{R^t}{r} \rceil$-subset of $[R]^t$ contains a t-dimensional m-box. Hence for large m, we get an inductive condition that bounds $P_t(m,r)$, and hence also $R_t(m,r)$:

$$\begin{aligned} P_1(m,r) &= (m-1)r+1 \\ P_t(m,r) &\leq \min_{x>r^m}\left\{\max\left(P_{t-1}(m,x), \tfrac{m^2 r}{2\log(x/r^m)}\right)\right\}. \end{aligned} \tag{8}$$

Since $R_t(m,r) \leq P_t(m,r)$ we get from (8) the following theorem:

Theorem 3.1. *For $r, t \geq 1$ we have*

$$\limsup_{m\in\boldsymbol{N}} \left\{\frac{R_t(m,r)}{m r^{m^{t-1}}}\right\} \leq 1.$$

Proof: We will prove that the statement of the theorem holds for $P_t(m,r)$. The theorem is clearly true when the dimension $t = 1$. We assume that $r, t \geq 2$, that m is sufficiently large and proceed by induction on t:

Assume $P_{t-1}(m,r) \leq a_{t-1}(m) r^{m^{t-2}}$ where $\lim_{m\to\infty} \frac{a_{t-1}(m)}{m} = 1$. From (8) we get for $t \geq 2$

$$P_t(m,r) \leq \min_{x>r^m}\left\{\max\left(a_{t-1}(m)x^{m^{t-2}}, \frac{m^2 r}{2\log(x/r^m)}\right)\right\}.$$

We note that $f(x) = a_{t-1}(m)x^{m^{t-2}}$ is increasing with $\lim_{x\to\infty} f(x) = \infty$ and $g(x) = \frac{m^2 r}{2\log(x/r^m)}$ is decreasing for $x > r^m$ with $\lim_{x\to\infty} g(x) = 0$ and $\lim_{x\to r^m+} g(x) = \infty$. Hence there is exactly one $x > r^m$ which would give the optimal bound for $P_t(m,r)$, namely the $x^* > r^m$ satisfying $f(x^*) = g(x^*)$. Let us examine this solution x^*:

If we let $y = x/r^m$ and hence $y \in]1;\infty[$, then the equation $f(x) = g(x)$ becomes

$$y^{m^{t-2}} \log y = \frac{m^2 r}{2a_{t-1}(m) r^{m^{t-1}}}. \tag{9}$$

We will show that the solution y^* is "close" to 1. Since $t \log t \geq t - 1$ for all $t > 0$ we have for all $y > 1$ that $\log y \geq 1 - \frac{1}{y} \geq 1 - \frac{1}{y^{m^{t-2}}}$ Hence the solution y' to the equation

$$y^{m^{t-2}} - 1 = y^{m^{t-2}} \left(1 - \frac{1}{y^{m^{t-2}}}\right) = \frac{m^2 r}{2a_{t-1}(m) r^{m^{t-1}}}$$

will be larger than the solution y^* of (9). Therefore $1 < y^* < y'$ and hence $r^m < x^* < r^m y'$. Since now $f(x)$ is an increasing function we get that

$$P_t(m, r) \leq f(x^*) < f(r^m y') = \left(a_{t-1}(m) + \frac{m^2 r}{2r^{m^{t-1}}}\right) r^{m^{t-1}}.$$

Letting $a_t(m) = a_{t-1}(m) + \frac{m^2 r}{2r^{m^{t-1}}}$ we get that $P_t(m,r) \leq a_t(m) r^{m^{t-1}}$ where $\lim_{m\to\infty} \frac{a_t(m)}{m} = 1$, which completes the induction and hence proves our theorem.$\square$

Remarks: (i) $mr^{m^{t-1}}$ is the best asymptotic bound that (8) can yield for $P_t(m,r)$. (ii) Looking at the inductive definition for $a_t(m)$ in the last paragraph of the above proof, we get that

$$a_t(m) - m = a_t(m) - a_1(m) = \sum_{i=1}^{t-1} (a_{i+1}(m) - a_i(m)) \leq \sum_{i=1}^{t-1} \frac{m^2 r}{2r^{m^i}}$$

and since $\sum_{i=1}^{t-1} \frac{1}{r^{m^i}} < \frac{1}{r^{m-1}} < \frac{2}{r^m}$ we get in particular that

$$R_t(m,r) \leq P_t(m,r) \leq \left(m + \frac{m^2}{r^{m-1}}\right) r^{m^{t-1}}$$

for sufficiently large m.

References

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