Extremal Graphs of Order Dimension 4

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Abstract

We will examine the maximal number of edges of a graph on p vertices of order dimension 4. We will show that this number is quadratic as a function of p, and we will show that asymptotically speaking it lies between $\frac{3}{8}p^2$ and $\frac{9}{20}p^2$.

1 Introduction

The maximal number, $e_3(p)$, of edges of a graph on p vertices of order dimension ≤ 3 can be determined easily by Schnyder's theorem [1, Thm (2.1) pg 128] stating that a graph has order dimension ≤ 3 if and only if it is planar, together with Eulers formula [2, Cor. 4.2.8], and hence get that $e_3(p) = 3p - 6$. One can ask the same question for graphs on p vertices of arbitrary order dimension. The first interesting and nontrivial case is considering graphs of order dimension 4, which is the main purpose of this paper. We will show that the maximal number, $e_4(p)$, of edges of a graph on p vertices of order dimension 4 is a quadratic function of p, which roughly lies between $\frac{3}{8}p^2$ and $\frac{9}{20}p^2$ for large p.

To determine the maximal number of edges of a graph on p vertices of order dimension n is the same as finding the maximal number of 1-faces of a simplicial complex on p vertices of order dimension n. Therefore we see that this problem is a special case of a more general one posed by Prof. Bernd Sturmfels of U. C. Berkeley: Determine how many k-faces a simplicial complex on p vertices of order dimension n can have. In order to solve this, it turns out that it suffices to study a very special kind of a simplicial complex, called the *Scarf complex* of a monomial ideal generated by p monomials in n variables, which we can assume to be *generic artinian* [3, Thm 6.5 pg 13]: A Scarf complex of a monomial ideal $M \subseteq k[X_1, \ldots, X_n]$ with minimal generators $W_1, \ldots, W_p$, is an abstract simplicial complex Δ_M on $\{1, \ldots, p\}$ of dimension $\leq n-1$, constructed in the following way: For a subset $S \subseteq \{1, \ldots, p\}$ let W_S be the least common multiple of $W_i, i \in S$. S is now a face of Δ_M if (i) for all $i \notin S$, W_i does not divide W_S, and (ii) for all $i \in S$, the support of W_S/W_i is a strict subset of $\{X_1, \ldots, X_n\}$. For a geometrical interpretation of this definition we note that the semigroup consisting of monomials of $k[X_1, \ldots, X_n]$ is isomorphic to the additive semigroup $\boldsymbol{N}_0^n$, via the isomorphism ϕ where $\phi(X_1^{i_1} \cdots X_n^{i_n}) = (i_1, \ldots, i_n)$. Let B_S be the n-dimensional hyper-box in $\boldsymbol{N}_0^n$ spanned by $\phi(W_S)$. S is now a face of Δ_M if B_S is the least hyper-box containing $\{\phi(W_i) : i \in S\}$ in its boundary (i.e. the union of the $(n-1)$-dimensional sides of B_S), and B_S contains no other $\phi(W_j)$. The ideal M is called generic artinian if $X_1^{d_1}, \ldots, X_n^{d_n}$ are among the generators for M, and no variable X_i appears with the same non-zero exponent in two distinct generators for M. The facets, or the maximal simplices of the Scarf complex Δ_M, are in 1-1 correspondence with the *outer corners* of the filter $\phi(M)$, which are the maximal points w.r.t. the component-wise partial order on $\boldsymbol{N}_0^n$, that are not contained in that filter.

Now let $c_n(p)$ be the maximal number of outer corners of any filter in $\boldsymbol{N}_0^n$ generated by p or fewer points. Since we soon will see that $c_4(p)$ and $e_4(p)$ are both quadratic in p, and only differ by a linear polynomial in p, we can view this present paper as a continuation of [4], where it is shown that the first nontrivial case for calculating $c_n(p)$ is for dimension $n = 4$, and where one of the main conclusions therein [4, Thm 35] is that for $p \geq 4$ we have $c_4(p) = \frac{p^2-3p-2}{2} - D(p)$ where $D : \boldsymbol{N}_4 = \boldsymbol{N}_0 \setminus \{0,1,2,3\} \to \boldsymbol{N}_0$ is a defect function satisfying $D(p) = 0$ for $p \in \{4, 5, \ldots, 12\}$, $D(13) = 1$, and $D(p) \geq 1$ for $p > 13$.

Convention: For a function $F : \boldsymbol{N}_0 \to \boldsymbol{N}_0$ let us denote

$$\liminf_{p \in \boldsymbol{N}_0} \{F(p)\} \geq a \text{ by } F(p) \succeq a \text{ and}$$
$$\limsup_{p \in \boldsymbol{N}_0} \{F(p)\} \leq b \text{ by } F(p) \preceq b.$$

We will show that the defect function $D : \boldsymbol{N}_4 \to \boldsymbol{N}_0$ from above satisfies $1/20 \preceq D(p)/p^2 \preceq 1/8$.

Remark: The fact that $D(13) = 1$ is the first case where $D(p)$ is nonzero led Prof. Sturmfels to believe that this might be related to Prof. William T. Trotter's result that the order dimension of the complete graph on 12 vertices, K_{12}, is 4, but the order dimension of K_{13} is 5 [5]. That he noted to be the case: Given an integer $p \geq 4$. We bear in mind the correspondence between filters in $\boldsymbol{N}_0^4$ and monomial ideals in four variables. That the order dimension of K_p is 4, is equivalent to the existence of a generic artinian monomial ideal M, generated by p elements in 4 variables, whose Scarf-complex Δ_M has precisely p vertices and $\binom{p}{2}$ edges [3, Thm 6.5, pg 13]. Looking at the corresponding simplicial Scarf-polytope P_M whose boundary is Δ_M and has therefore also $f_1(P_M) = \binom{p}{2}$ edges, we get by the Dehn-Sommerville equations [6, pg 252] that the number of facets of P_M is $f_3(P_M) = f_1(P_M) - p = \frac{p^2-3p}{2}$, and hence the outer corner points of M are $f_3(P_M) - 1 = \frac{p^2-3p-2}{2}$. Since now the outer corner points of M are $\leq c_4(p) = \frac{p^2-3p-2}{2} - D(p)$ we have that $D(p) = 0$ and hence $p \leq 12$.

By carrying on in the setup of the above remark, we have in fact for arbitrary $p \geq 4$, that a generic artinian monomial ideal M in 4 variables generated by p monomials with $c_4(p) = \frac{p^2-3p-2}{2} - D(p)$ outer corner points, that the Scarf polytope P_M has $f_3(P_M) = c_4(p) + 1$ facets, and hence by the Dehn-Sommerville equations has $f_1(P_M) = f_3(P_M) + p = \frac{p^2-3p-2}{2} - D(p) + 1 + p = \binom{p}{2} - D(p)$ edges. Hence one can therefore interpret the defect function $D : \boldsymbol{N}_4 \to \boldsymbol{N}_0$ as follows:

Proposition 1.1 *The maximal number, $e_4(p)$, of edges of a graph on p vertices which has order dimension 4 is $\binom{p}{2} - D(p)$.*$\Box$

Let us derive an upper bound for $D : \boldsymbol{N}_4 \to \boldsymbol{N}_0$. By using an easy product property of filters of $\boldsymbol{N}_0^4$ and $\boldsymbol{N}_0^2$ [4, Prop. 5], we establish that $c_4(2q) \geq c_2(q)^2$ for all $q \geq 2$. Since $c_2(q) = q - 1$ we have that $c_4(p)/p^2 \succeq 1/4$. We have in fact a better result. For a real number x let $\lfloor x \rfloor$ denote the largest integer $\leq x$, and let $\lceil x \rceil$ denote the smallest integer $\geq x$. The following theorem yields that $c_4(p)/p^2 \succeq 3/8$.

Theorem 1.2 *For $p \geq 8$ we have*

$$c_4(p) \geq \frac{p^2 + 3p - 26}{2} - \left(\left\lfloor \frac{p}{4} \right\rfloor + 1 \right) \left(p - 2 \left\lfloor \frac{p}{4} \right\rfloor \right).$$

Sketch of proof: Note that the statement of this theorem is equivalent to

$$\begin{array}{ll} c_4(4q) & \geq 6q^2+4q-13, \\ c_4(4q+1) & \geq 6q^2+7q-12, \\ c_4(4q+2) & \geq 6q^2+10q-10 \text{ and} \\ c_4(4q+3) & \geq 6q^2+13q-7, \text{ for } q \geq 2. \end{array}$$

Hence it suffices to provide examples of filters generated by $4q, 4q+1, 4q+2$ and $4q+3$ points in $\boldsymbol{N}_0^4$ that have $6q^2+4q-13, 6q^2+7q-12, 6q^2+10q-10$ and $6q^2+13q-7$ outer corner points respectively.

A filter U_{4q} generated by $4q$ points:

$$\begin{array}{llc} (4q-3,0,0,0), & (3q-2-i, 3q-3+i, 2q-1-i, i) & i=1,\dots,q-1, \\ (0,4q-3,0,0), & (3q-3+i, 3q-2-i, i, 2q-1-i) & -, \\ (0,0,4q-3,0), & (i, 2q-1-i, 3q-3+i, 3q-2-i) & -, \\ (0,0,0,4q-3), & (2q-1-i, i, 3q-2-i, 3q-3+i) & -. \end{array}$$

A filter U_{4q+1} generated by $4q+1$ points:

$$\begin{array}{llc} (4q-2,0,0,0), & (3q-1-i, 3q-3+i, 2q-i, i) & i=1,\dots,q, \\ (0,4q-2,0,0), & (3q-2+i, 3q-2-i, i, 2q-i) & i=1,\dots,q-1, \\ (0,0,4q-2,0), & (i, 2q-1-i, 3q-2+i, 3q-1-i) & -, \\ (0,0,0,4q-2), & (2q-1-i, i, 3q-1-i, 3q-2+i) & -. \end{array}$$

A filter U_{4q+2} generated by $4q+2$ points:

$$\begin{array}{llc} (4q-1,0,0,0), & (3q-1-i, 3q-2+i, 2q+1-i, i) & i=1,\dots,q, \\ (0,4q-1,0,0), & (3q-2+i, 3q-1-i, i, 2q+1-i) & -, \\ (0,0,4q-1,0), & (i, 2q-1-i, 3q-1+i, 3q-i) & i=1,\dots,q-1, \\ (0,0,0,4q-1), & (2q-1-i, i, 3q-i, 3q-1+i) & -. \end{array}$$

A filter U_{4q+3} generated by $4q+3$ points:

$$\begin{array}{llc} (4q,0,0,0), & (3q-i, 3q-1+i, 2q+1-i, i) & i=1,\dots,q, \\ (0,4q,0,0), & (3q-1+i, 3q-i, i, 2q+1-i) & -, \\ (0,0,4q,0), & (i, 2q-i, 3q-1+i, 3q+1-i) & -, \\ (0,0,0,4q), & (2q-i, i, 3q-i, 3q+i) & i=1,\dots,q-1. \end{array}$$

These filters, $U_{4q}, U_{4q+1}, U_{4q+2}$ and U_{4q+3}, have $6q^2+4q-13, 6q^2+7q-12, 6q^2+10q-10$ and $6q^2+13q-7$ outer corner points respectively.$\square$

REMARK: If U is a filter in $\boldsymbol{N}_0^4$ let I_U be the unique corresponding monomial ideal of $k[X_1, X_2, X_3, X_4]$, and $\mathbf{m}$ be the maximal ideal generated by the four indeterminates, then the number of the outer corner points of U is $\dim_k((\mathbf{m}:I_U)/I_U)$, which can be calculated in each case using a computer algebra system i.e. [7].

The above theorem implies that $D(p) \le \left(\lfloor \frac{p}{4} \rfloor + 1\right)\left(p - 2\lfloor \frac{p}{4} \rfloor\right) - 3p + 12$, and hence $D(p)/p^2 \preceq 1/8$.

2 A quadratic lower bound for $D(p)$

We will start this section by recalling some notations, useful definitions and facts from [4]:

- For a filter $U \subseteq \boldsymbol{N}_0^n$, $\mathrm{Out}(U)$ is the set of *outer corner points* of U, that is the maximal elements of $\boldsymbol{N}_0^n \setminus U$ under the component-wise partial order. $\le$ will always mean that partial order unless otherwise stated.
- $\mathcal{C}_p^n$ is the set of filters of $\boldsymbol{N}_0^n$ that are generated by p or fewer points.
- $\tilde{x} \in \boldsymbol{N}_0^n$ is *i-adjacent* to a filter U iff $\tilde{x} \notin U$ and $\tilde{x} + \tilde{e}_i \in U$. $\tilde{x} \in \boldsymbol{N}_0^n$ is *adjacent* to U if $\tilde{x}$ is i-adjacent to U for some $i \in \{1, \ldots, n\}$.
- For $\tilde{x} \in \boldsymbol{N}_0^n$, $P_i(\tilde{x})$ denotes the i-th component x_i of $\tilde{x}$, and $\pi_{i_1 \ldots i_k} : \boldsymbol{N}_0^n \to \boldsymbol{N}_0^{n-k}$ is the projection where components $i_1, \ldots, i_k$ are omitted. π will throughout mean the map $\pi_4 : \boldsymbol{N}_0^4 \to \boldsymbol{N}_0^3$.
- The filter of $\boldsymbol{N}_0^n$ generated by a singleton $\tilde{m}$ is denoted by $C(\tilde{m})$ and called an *orthant*. In the case $n = 3$, $C(\tilde{m})$ is called an *octant*.

Note the following convenient characterization of outer corners of a filter $U \subseteq \boldsymbol{N}_0^n$: $\tilde{x} \in \mathrm{Out}(U)$ if and only if $\tilde{x} \notin U$ and $\tilde{x}$ is i-adjacent to U for all $i = 1, \ldots, n$. Recall a useful lemma from [4, Lemma 24]:

Lemma 2.1 *For a filter U in $\boldsymbol{N}_0^3$ generated by p elements there are at most $p-1$ points of $\mathrm{Out}(U)$ that are adjacent to the octant $C(\tilde{m})$, where $\tilde{m}$ is one of the generators for U.*$\square$

Definition 2.2 *Let U be a filter generated by $\tilde{m}_1, \ldots, \tilde{m}_p \in \boldsymbol{N}_0^n$.*

1. *We say that U is* generic *if the nonzero integers of $P_i(\tilde{m}_l)$ $l = 1, \ldots, p$ are distinct, for all $i = 1, \ldots, n$.*

2. *We say that U is* artinian *if $\boldsymbol{N}_0^n \setminus U$ is a finite set, that is $p \geq n$ and n of the generators for U have the form $k_i\tilde{e}_i$, $i = 1, \ldots, n$.*

3. *We say U is* condensed *if*

$$\begin{aligned}
&\tilde{m}_l = (p-n+1)\tilde{e}_l \text{ for } l \in \{1, \ldots, n-1\}, \\
&\tilde{m}_p = (p-n+1)\tilde{e}_n, \\
&P_n(\tilde{m}_l) = l - n + 1 \text{ for } l \in \{n, \ldots, p-1\}, \\
&\{P_i(\tilde{m}_l) : l = n, \ldots, p-1\} = \{1, \ldots, p-n\} \text{ for } i \in \{1, \ldots, n-1\}.
\end{aligned}$$

Note that a condensed filter is generic artinian. As shown in [4, Prop. 20] there are condensed filters $U \in \mathcal{C}_p^4$ such that $|\mathrm{Out}(U)| = c_4(p)$. Let U be such, and assume it is generated by $\tilde{m}_1, \ldots, \tilde{m}_p$. For $i \in \{2, 3, \ldots, p\}$ let U_i be the filter generated by the first i points $\tilde{m}_1, \ldots, \tilde{m}_i$. Since $\tilde{m}_p = (p-n+1)\tilde{e}_n$ we have from the proof lemma 15 in [4] that $|\mathrm{Out}(U)| = |\mathrm{Out}(U_{p-1})| + |\mathrm{Out}(\pi(U_{p-1}))|$ and hence by lemma 10(iii) and corollary 13 of [4] we have:

$$c_4(p) = \sum_{i=3}^{p-1} |\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1}))| \tag{1}$$

Note that for general filters we have $\geq$ instead of equality in (1). Since $\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1}))$ consists of points of $\mathrm{Out}(\pi(U_i))$ that are adjacent to $C(\pi(\tilde{m}_i))$, we see that $|\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1}))|$ is equal to 1 for $i = 3$ and by lemma 2.1 is $\leq i - 1$ for $i \geq 4$. The defect function has therefore the following form

$$D(p) = \sum_{i=4}^{p-1} (i - 1 - |\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1}))|) \tag{2}$$

where each term on the right is nonnegative. Again note that for general filters we have $\leq$ instead of equality in (2).

For a condensed filter $U \in \mathcal{C}_p^4$ let $M = \{\tilde{m}_4, \ldots, \tilde{m}_{p-1}\} \subseteq \boldsymbol{N}_0^4$ be the set of generators of U that have no zeros in their coordinates. We have then:

Lemma 2.3 *If $\pi_{j4}(M) \subseteq \boldsymbol{N}_0^2$ contains a q-chain for a condensed filter U of $\boldsymbol{N}_0^4$ for some $j \in \{1, 2, 3\}$, then for some $i \in \{q+3, \ldots, p-1\}$ we have*

$$|\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1}))| \leq i - q + 1. \tag{3}$$

REMARKS: Note that every point of $\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1}))$ is adjacent to the octant $C(\pi(\tilde{m}_i))$ and two other generating octants of U_i in $\boldsymbol{N}_0^3$. Hence one can say that (3) means that $C(\pi(\tilde{m}_i))$ fails to form outer corner points with $q-2$ of the other octants $C(\pi(\tilde{m}_l))$, $l < i$.

Clearly one can deduce that if $\pi_{jk}(M)$ contains a q-chain in $\boldsymbol{N}_0^2$ for some distinct $j, k \in \{1,2,3,4\}$ then for $h \in \{j,k\}$ and some for some $i \in \{q+3,\ldots,p\}$ $C(\pi_h(\tilde{m}_i))$ fails to form outer corner points with $q-2$ of the octants $C(\pi_h(\tilde{m}_l))$ $l < i$.

Before proving lemma 2.3 we state a useful fact:

Remark 2.4 *If $\tilde{a}, \tilde{b}, \tilde{c} \in \boldsymbol{N}_0^3$ are such that for some $l \in \{1,2,3\}$*

$$\pi_l(\tilde{a}) > \pi_l(\tilde{b})$$
$$P_l(\tilde{a}) < P_l(\tilde{b}) < P_l(\tilde{c})$$

then all the points adjacent to both $C(\tilde{a})$ and $C(\tilde{c})$, if any, are contained in $C(\tilde{b})$ and hence if $\tilde{a}, \tilde{b}, \tilde{c}$ are among generators for a filter U of $\boldsymbol{N}_0^3$ then no point of $\mathrm{Out}(U)$ is adjacent to both $C(\tilde{a})$ and $C(\tilde{c})$.$\square$

PROOF OF LEMMA 2.3: For simplicity we may assume $j = 3$. Let $Q \subseteq M$ be such that $\pi_{34}(Q) \subseteq \boldsymbol{N}_0^2$ is the q-chain in question. We will show that for i where $\tilde{m}_i \in Q$ has the largest index (that is, largest last coordinate) we have (3).

We have two cases to consider:

$\pi_{14}(\tilde{m}_i)$ is not a minimal element of $\pi_{14}(Q)$: Here $\pi_{14}(\tilde{m}_i) > \pi_{14}(\tilde{m}_j)$ for some $\tilde{m}_j \in Q$. By our assumption on $\tilde{m}_i$ and the fact that $\pi_{34}(Q)$ is a q-chain in $\boldsymbol{N}_0^2$ we have that $\tilde{m}_i > \tilde{m}_j$ in $\boldsymbol{N}_0^4$ and hence in particular $\pi(U_i) = \pi(U_{i-1})$ and therefore $\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1})) = \emptyset$ proving the lemma in this case.

$\pi_{14}(\tilde{m}_i)$ is a minimal element of $\pi_{14}(Q)$: Write Q as a disjoint union $Q_{\min} \cup Q'$ where $\pi_{14}(Q_{\min})$ are the minimal elements of $\pi_{14}(Q)$. Since $\pi_{34}(Q)$ is a q-chain in $\boldsymbol{N}_0^2$ we have for any $\tilde{m}_j \in Q'$ that there is a $\tilde{m}_k \in Q_{\min}$, $k \leq i$, such that $\pi(\tilde{m}_j) > \pi(\tilde{m}_k)$ and hence if U_i' is the filter generated all the first i points except the ones in Q' then $\pi(U_i) = \pi(U_i')$ and so $\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1})) \subseteq \mathrm{Out}(\pi(U_i')) \setminus \mathrm{Out}(\pi(U_{i-1}))$. Note that every point in this set must be adjacent to $C(\pi(\tilde{m}_i))$ since otherwise it would be in $\mathrm{Out}(\pi(U_{i-1}))$. By lemma 2.1 there can only be $i - |Q'| - 1$ such points. If $q - |Q'| \leq 2$ then we are done, so assume $q - |Q'| \geq 3$.

Let $r = q - |Q'|$. Since $\pi_{34}(Q_{\min})$ is an r-chain and $\pi_{14}(Q_{\min})$ a r-antichain in $\boldsymbol{N}_0^2$ we can write $Q_{\min} = \{\tilde{m}'_1, \ldots, \tilde{m}'_r\}$ where

$$\begin{array}{l} \pi_{34}(\tilde{m}'_1) < \pi_{34}(\tilde{m}'_2) < \cdots < \pi_{34}(\tilde{m}'_r) \\ P_3(\tilde{m}'_1) > P_3(\tilde{m}'_2) > \cdots > P_3(\tilde{m}'_r). \end{array} \tag{4}$$

If $\tilde{m}_i = \tilde{m}'_\alpha$ where $\alpha \in \{1, r\}$, let $\alpha' = \alpha + (-1)^{\lfloor \frac{\alpha}{r} \rfloor}$. Considering $\pi(\tilde{m}'_\alpha)$, $\pi(\tilde{m}'_{\alpha'})$ and $\pi(\tilde{m}'_l)$ where $l \in \{1, \ldots, r\} \setminus \{\alpha, \alpha'\}$ we have by remark 2.4 that there are no points in $\mathrm{Out}(\pi(U_i))$ adjacent to both $C(\pi(\tilde{m}_i)) = C(\pi(\tilde{m}'_\alpha))$ and $C(\pi(\tilde{m}'_l))$. Hence all the points in $\mathrm{Out}(\pi(U_i))$ adjacent to $C(\pi(\tilde{m}_i))$ must be contained in $\mathrm{Out}(\pi(U_i^{(1)}))$ where $U_i^{(1)}$ is the filter generated by the first i points $\tilde{m}_1, \ldots, \tilde{m}_i$ except the ones in $Q' \cup (Q_{\min} \setminus \{\tilde{m}'_\alpha, \tilde{m}'_{\alpha'}\})$. Therefore $\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1})) \subseteq \mathrm{Out}(\pi(U_i^{(1)})) \setminus \mathrm{Out}(\pi(U_{i-1}))$ which is contained in the set of points of $\mathrm{Out}(\pi(U_i^{(1)}))$ adjacent to $C(\pi(\tilde{m}_i))$. Since $|Q' \cup (Q_{\min} \setminus \{\tilde{m}'_\alpha, \tilde{m}'_{\alpha'}\})| = q - 2$ there are by lemma 2.1 at most $i - q + 1$ such points, proving the lemma in this case.

Finally consider the case where $\tilde{m}_i = \tilde{m}'_\beta$ where $\beta \in \{2, 3, \ldots, r-1\}$. Recall that since U is condensed $\tilde{m}_3 = (0, 0, p-3, 0)$ and hence $\pi_{34}(\tilde{m}_3) < \pi_{34}(\tilde{m}'_1)$ and $P_3(\tilde{m}_3) > P_3(\tilde{m}'_1)$. Let $\tilde{m}'_0 = \tilde{m}_3$. This together with (4) yields that there are no points in $\mathrm{Out}(\pi(U_i))$ adjacent to both $C(\pi(\tilde{m}_i)) = C(\pi(\tilde{m}'_\beta))$ and $C(\pi(\tilde{m}'_l))$ where $l \in \{0, \ldots, \beta - 2, \beta + 2, \ldots, r\}$. (Note that for $\beta = r - 1$ the string $\beta + 2, \ldots, r$ is empty.) Hence all the points in $\mathrm{Out}(\pi(U_i))$ adjacent to $C(\pi(\tilde{m}_i))$ must be contained in $\mathrm{Out}(\pi(U_i^{(2)}))$ where $U_i^{(2)}$ is the filter generated by all the points $\tilde{m}_1, \ldots, \tilde{m}_i$ except the ones in $Q' \cup (Q_{\min} \setminus \{\tilde{m}'_{\beta-1}, \tilde{m}'_\beta, \tilde{m}'_{\beta+1}\}) \cup \{\tilde{m}_3\}$. Hence we have $\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1})) \subseteq \mathrm{Out}(\pi(U_i^{(2)})) \setminus \mathrm{Out}(\pi(U_{i-1}))$ which by lemma 2.1 contains at most $i - q + 1$ points, proving the lemma in this final case.□

We can now use the above lemma to prove:

Lemma 2.5 *If $\pi_{j4}(M) \subseteq \boldsymbol{N}_0^2$ contains a $3(q-1)$-antichain for a condensed filter U of $\boldsymbol{N}_0^4$ for some $j \in \{1, 2, 3\}$, then for some $i \in \{q+3, \ldots, p-1\}$ we have (3), that is*

$$|\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1}))| \leq i - q + 1.$$

PROOF: We may assume $j = 3$. Let $A \subseteq M$ be such that $\pi_{34}(A) \subseteq \boldsymbol{N}_0^2$ is the $3(q-1)$-antichain in question. We consider that i where $\tilde{m}_i \in A$ has the highest index. Now write A as a disjoint union $A = \{\tilde{m}_i\} \cup B \cup C$ where

$$\begin{aligned} B &= \{\tilde{m}_l \in A : P_1(\tilde{m}_l) > P_1(\tilde{m}_i)\} \\ C &= \{\tilde{m}_l \in A : P_1(\tilde{m}_l) < P_1(\tilde{m}_i)\} \end{aligned}$$

write B and C as a disjoint union, $B = B_{\min} \cup B'$ and $C = C_{\min} \cup C'$, where $\pi_{24}(B_{\min})$ are the minimal elements of $\pi_{24}(B)$ and $\pi_{14}(C_{\min})$ the minimal elements of $\pi_{14}(C)$. Note that $\pi_{14}(B_{\min})$ and $\pi_{24}(C_{\min})$ are both chains in $\boldsymbol{N}_0^2$.

Consider an element $\tilde{m}_\beta \in B'$. Since $\pi_{24}(\tilde{m}_\beta)$ is not a minimal element of $\pi_{24}(B)$ there is a $\tilde{m}_\gamma \in B_{\min}$ such that $\pi_{24}(\tilde{m}_\gamma) < \pi_{24}(\tilde{m}_\beta)$. By the mere fact that $\pi_{34}(A)$ is an antichain in $\boldsymbol{N}_0^2$ we have for $\tilde{m}_i, \tilde{m}_\beta, \tilde{m}_\gamma$ that

$$\begin{gathered} \pi_{24}(\tilde{m}_\gamma) < \pi_{24}(\tilde{m}_\beta), \\ P_2(\tilde{m}_i) > P_2(\tilde{m}_\gamma) > P_2(\tilde{m}_\beta) \end{gathered}$$

so by remark 2.4 there is no point of $\mathrm{Out}(\pi(U_i))$ adjacent to both $C(\pi(\tilde{m}_i))$ and $C(\pi(\tilde{m}_\beta))$. Similarly if $\tilde{m}_\nu \in C'$ then no point in $\mathrm{Out}(\pi(U_i))$ adjacent to both $C(\pi(\tilde{m}_i))$ and $C(\pi(\tilde{m}_\nu))$. Hence all the points in $\mathrm{Out}(\pi(U_i))$ adjacent to $C(\pi(\tilde{m}_i))$ must be contained in $\mathrm{Out}(\pi(U_i^!))$ where $U_i^!$ is the filter generated by all the points $\tilde{m}_1, \ldots, \tilde{m}_i$ except the ones in $B' \cup C'$. So $\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1})) \subseteq \mathrm{Out}(\pi(U_i^!)) \setminus \mathrm{Out}(\pi(U_{i-1}))$ which is contained in the set of points of $\mathrm{Out}(\pi(U_i^!))$ adjacent to $C(\pi(\tilde{m}_i))$. By lemma 2.1 there are at most $i - |B' \cup C'| - 1$ such points.

Now, since $\pi_{34}(A), \pi_{24}(B_{\min})$ and $\pi_{14}(C_{\min})$ are all antichains in $\boldsymbol{N}_0^2$ and $B_{\min} \cup C_{\min} \subseteq A$ we see that both $\pi_{14}(B_{\min})$ and $\pi_{24}(C_{\min})$ are chains in $\boldsymbol{N}_0^2$, and hence if either $|B_{\min}|$ or $|C_{\min}|$ is $\geq q$ then by lemma 2.3 we are done. So we may assume both $|B_{\min}|$ and $|C_{\min}|$ to be $\leq q-1$. By our assumptions in the lemma we have $3(q-1) = |A| = 1 + |B' \cup C'| + |B_{\min}| + |C_{\min}| \leq 1 + |B' \cup C'| + 2(q-1)$ and therefore $|B' \cup C'| \geq q-2$. Hence $i - |B' \cup C'| - 1$ from above, is $\leq i - (q-2) - 1 = i - q + 1$ proving the lemma.□

We can now prove the following:

Lemma 2.6 *If a condensed filter U is generated by $p \geq 10q$ elements then for some $i \in \{q+3, \ldots, p-1\}$ we have (3), that is*

$$|\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1}))| \leq i - q + 1.$$

PROOF: We start with an observation. If $\pi(\tilde{m}_{p-1}) > \pi(\tilde{m}_l)$ for some $l < p-1$ then $\pi(U_{p-1}) = \pi(U_{p-2})$ and hence $\mathrm{Out}(\pi(U_{p-1})) \setminus \mathrm{Out}(\pi(U_{p-2})) = \emptyset$ proving the theorem in this case, so we may assume $\pi(\tilde{m}_{p-1})$ to be a minimal element of $\pi(M)$ in $\boldsymbol{N}_0^3$. Write M as a disjoint union

$$M = A \cup \{\tilde{m}_{p-1}\} \cup B_1 \cup B_2 \cup B_3 \cup C_1 \cup C_2 \cup C_3$$

where

$$\begin{aligned}
A &= \{\tilde{m} \in M : \pi(\tilde{m}) > \pi(\tilde{m}_{p-1})\} \\
B_l &= \{\tilde{m} \in M : \pi_{l4}(\tilde{m}) > \pi_{l4}(\tilde{m}_{p-1}) \text{ and } P_l(\tilde{m}) < P_l(\tilde{m}_{p-1})\} \\
C_l &= \{\tilde{m} \in M : \pi_{l4}(\tilde{m}) < \pi_{l4}(\tilde{m}_{p-1}) \text{ and } P_l(\tilde{m}) > P_l(\tilde{m}_{p-1})\}.
\end{aligned}$$

For each $l \in \{1,2,3\}$ write B_l and C_l as a disjoint union

$$\begin{aligned}
B_l &= B_l^{\min} \cup B_l' \\
C_l &= C_l^{\min} \cup C_l'
\end{aligned}$$

where $\pi_{l4}(B_l^{\min})$ are the minimal elements of $\pi_{l4}(B_l)$ and $P_l(C_l^{\min})$ is the set of minimal elements of $P_l(C_l)$. (Needless to say, $C_l^{\min}$ has at most one element).

Clearly if $\tilde{m} \in A$ then there is no point of $\mathrm{Out}(\pi(U_{p-1}))$ adjacent to both $C(\pi(\tilde{m}_{p-1}))$ and $C(\pi(\tilde{m}))$.

If $\tilde{m} \in B_l'$ for some $l \in \{1,2,3\}$ then there is a $\tilde{m}^o \in B_l^{\min}$ such that $\pi_{l4}(\tilde{m}^o) < \pi_{l4}(\tilde{m})$. If now $P_l(\tilde{m}^o) < P_l(\tilde{m})$ then $\pi(\tilde{m}^o) < \pi(\tilde{m})$ and hence all the points adjacent to $C(\pi(\tilde{m}))$ are contained in $C(\pi(\tilde{m}^o))$. If on the other hand $P_l(\tilde{m}^o) > P_l(\tilde{m})$ then by remark 2.4 all the points adjacent to both $C(\pi(\tilde{m}_{p-1}))$ and $C(\pi(\tilde{m}))$ are contained in $C(\pi(\tilde{m}^o))$.

Finally if $\tilde{m} \in C_l'$ for some $l \in \{1,2,3\}$ then $P_l(\tilde{m}') < P_l(\tilde{m})$ for the unique element $\tilde{m}' \in C_l^{\min}$ so by remark 2.4 all the points adjacent to both $C(\pi(\tilde{m}_{p-1}))$ and $C(\pi(\tilde{m}))$ are contained in $C(\pi(\tilde{m}'))$.

In all the three cases from above we can see there is no point in $\mathrm{Out}(\pi(U_{p-1}))$ that is adjacent to both $C(\pi(\tilde{m}_{p-1}))$ and $C(\pi(\tilde{m}))$, and hence all the points in $\mathrm{Out}(\pi(U_{p-1}))$ adjacent to $C(\pi(\tilde{m}_{p-1}))$ are contained in $\mathrm{Out}(\pi(U_{p-1}'))$ where U_{p-1}' is the filter generated by all the $\tilde{m}_i, i = 1, \dots, p-1$ except the ones in $S = A \cup B_1' \cup B_2' \cup B_3' \cup C_1' \cup C_2' \cup C_3'$. So $\mathrm{Out}(\pi(U_{p-1})) \setminus \mathrm{Out}(\pi(U_{p-2})) \subseteq \mathrm{Out}(\pi(U_{p-1}')) \setminus \mathrm{Out}(\pi(U_{p-2}))$ which is contained in the set of points of $\mathrm{Out}(\pi(U_{p-1}'))$ adjacent to $C(\pi(\tilde{m}_{p-1}))$. For $i = p-1$ there are by lemma 2.1 at most $i - |S| - 1$ such points.

Now, note that for each $l \in \{1,2,3\}$ $B_l^{\min} \subseteq M$ and $\pi_{l4}(B_l^{\min})$ is an antichain in $\boldsymbol{N}_0^2$, so by lemma 2.5 we may assume for the rest of this proof that $|B_l^{\min}| \leq 3q-4$, since otherwise we are done. Also note that each $C_l^{\min}$ has at most one element. Hence we have

$$
\begin{aligned}
& p-4 \\
= \; & |M| \\
= \; & |S \cup \{\tilde{m}_{p-1}\} \cup B_1^{\min} \cup B_2^{\min} \cup B_3^{\min} \cup C_1^{\min} \cup C_2^{\min} \cup C_3^{\min}| \\
\leq \; & |S| + 1 + 3(3q-4) + 3
\end{aligned}
$$

So $|S| \geq p-9q+4$. We have therefore finally that $i-|S|-1 \leq i-(p-9q+4) < i-q+1$ proving the theorem.□

Before we display one of our main results we need a technical lemma:

Lemma 2.7 *Let U be a generic artinian filter of $\boldsymbol{N}_0^3$ with $\tilde{m}_1$ and $\tilde{m}_2$ as two of its generators, neither one of the form $k_l\tilde{e}_l, l \in \{1,2,3\}$. Let*

$U_1 =$ *the filter generated by all the generators of U except $\tilde{m}_1$,*
$U_2 =$ *the filter generated by all the generators of U except $\tilde{m}_2$,*
$U_{12} =$ *the filter generated by all the generators of U except $\tilde{m}_1$ and $\tilde{m}_2$.*

Then we have that $|\mathrm{Out}(U_2) \setminus \mathrm{Out}(U_{12})| + 1 \geq |\mathrm{Out}(U) \setminus \mathrm{Out}(U_1)|$.

PROOF: Let us first dispatch the case where $\tilde{m}_1$ and $\tilde{m}_2$ are comparable in $\boldsymbol{N}_0^3$. If $\tilde{m}_1 \geq \tilde{m}_2$ then clearly $U = U_1$ and hence $\mathrm{Out}(U) \setminus \mathrm{Out}(U_1) = \emptyset$ proving the lemma in this case. If $\tilde{m}_1 \leq \tilde{m}_2$ then $U = U_2$ so $\mathrm{Out}(U) \setminus \mathrm{Out}(U_1) = \mathrm{Out}(U_2) \setminus \mathrm{Out}(U_{12})$ are the points of $\mathrm{Out}(U)$ adjacent to $C(\tilde{m}_1)$, so the lemma follows.

For the rest of this proof we will therefore assume $\tilde{m}_1$ and $\tilde{m}_2$ to be uncomparable in $\boldsymbol{N}_0^3$. We have two cases to consider, the first case where $P_l(\tilde{m}_1) < P_l(\tilde{m}_2)$ and $\pi_l(\tilde{m}_1) > \pi_l(\tilde{m}_2)$ for some $l \in \{1,2,3\}$, and the second case where $P_l(\tilde{m}_1) > P_l(\tilde{m}_2)$ and $\pi_l(\tilde{m}_1) < \pi_l(\tilde{m}_2)$ for some $l \in \{1,2,3\}$. Without loss of generality we may assume l to be 1 in both cases. Consider all the points in $\mathrm{Out}(U) \setminus \mathrm{Out}(U_1)$ that are not adjacent to $C(\tilde{m}_2)$ and call that set N. This set must be contained in $(\mathrm{Out}(U_2) \setminus \mathrm{Out}(U_{12})) \cap \mathrm{Out}(U)$. Let K be the set of points of $\mathrm{Out}(U) \setminus \mathrm{Out}(U_1)$ that are adjacent to $C(\tilde{m}_2)$.

First case: Here each point in K is 1-adjacent to $C(\tilde{m}_2)$ and either 2 or 3-adjacent to $C(\tilde{m}_1)$. Assume $\tilde{x} \in K$ is 1-adjacent to $C(\tilde{m}_2)$ and 2-adjacent to $C(\tilde{m}_1)$. Now let $\tilde{m}_3$ be the generator of U such that $\tilde{x}$ is 3-adjacent to $C(\tilde{m}_3)$. We see here that $\tilde{x}$ is uniquely determined by $\tilde{x} = (P_1(\tilde{m}_2) - 1, P_2(\tilde{m}_1) - 1, P_3(\tilde{m}_3) - 1)$. Note that if $\tilde{m}$ is a generator of U other than $\tilde{m}_2$ satisfying $\pi_1(\tilde{m}) < \pi_1(\tilde{x})$, then we have $P_1(\tilde{m}_2) < P_1(\tilde{m})$. Now let $\tilde{m}_2'$ be the generator of U_2 such that $\pi_1(\tilde{m}_2') < \pi_1(\tilde{x})$ and such that $P_1(\tilde{m}_2')$ is minimal. Since $k_1\tilde{e}_1$ is one of the generators for U_2 we know that such $\tilde{m}_2'$ exists. Let $\tilde{x}' = (P_1(\tilde{m}_2') - 1, P_2(\tilde{m}_1) - 1, P_3(\tilde{m}_3) - 1)$. By our definition of $\tilde{m}_2'$ we have $\tilde{x}' \notin U_2$ and also by the fact that U is generic artinian we have $P_1(\tilde{m}_2) < P_1(\tilde{m}_2')$ and hence $\tilde{x}' \in C(\tilde{m}_2) \subseteq U$, and therefore $\tilde{x}' \in U \setminus U_2$. Because $\tilde{x}'$ is 2-adjacent to $C(\tilde{m}_1)$ it cannot be 2-adjacent to any other generator of U since U is generic artinian. Hence $\tilde{x}' \notin \mathrm{Out}(U_{12})$. Clearly $\tilde{x}'$ is 1,2 and 3-adjacent to U_2 and we have therefore $\tilde{x}' \in (\mathrm{Out}(U_2) \setminus \mathrm{Out}(U_{12})) \setminus \mathrm{Out}(U)$. Similarly if $\tilde{y} \in K$ is 1-adjacent to $C(\tilde{m}_2)$ and 3-adjacent to $C(\tilde{m}_1)$ we get that $\tilde{y}$ is uniquely determined and we get a corresponding uniquely determined $\tilde{y}' \in (\mathrm{Out}(U_2) \setminus \mathrm{Out}(U_{12})) \setminus \mathrm{Out}(U)$. Therefore K consists of at most two points, $\tilde{x}$ and $\tilde{y}$. Since $\tilde{x}'$ is 2-adjacent to $C(\tilde{m}_1)$ and $\tilde{y}'$ is 3-adjacent to $C(\tilde{m}_1)$, $\tilde{x}'$ and $\tilde{y}'$ are distinct, so for $K' = \{\tilde{x}', \tilde{y}'\}$ we have $|K'| = |K|$.

Second case: Here each point in K is 1-adjacent to $C(\tilde{m}_1)$ and either 2 or 3-adjacent to $C(\tilde{m}_2)$. By the same argument as in the first case we get that K consists of at most two points, $\tilde{x}$ and $\tilde{y}$, but unlike the first case, their corresponding points, $\tilde{x}'$ and $\tilde{y}'$, might coincide in this case. Hence, all we can conclude here is that $|K'| + 1 \geq |K|$, which is in fact also valid in the first case. Hence in both cases we have

$$\begin{aligned} & |\mathrm{Out}(U) \setminus \mathrm{Out}(U_1)| \\ = \; & |N \cup K| \\ \leq \; & |N| + |K'| + 1 \\ = \; & |N \cup K'| + 1 \\ \leq \; & |\mathrm{Out}(U_2) \setminus \mathrm{Out}(U_{12})| + 1 \end{aligned}$$

which completes our proof.□

We can now use the last two lemmas to prove our main theorem:

Theorem 2.8 *If for $p \geq 4$ and $q = \left\lfloor \frac{p-1}{10} \right\rfloor$, then the defect function $D : \boldsymbol{N}_4 \to \boldsymbol{N}_0$ satisfies*

$$D(p) \geq (q-2)(p - 5q - 15). \tag{5}$$

Proof: We will use induction on p. Since $D(p) \geq 0$ for all p we have (5) for $4 \leq p \leq 30$.

Assume $p > 30$. Let U be a condensed filter generated by $\tilde{m}_1, \ldots, \tilde{m}_p$, with $c_4(p)$ outer corners. We have $D(p) = \sum_{i=4}^{p-1} D_i$, where $D_i = i - 1 - |\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1}))|$. Since $p - 1 \geq 10q$ there is by lemma 2.6 an $j \in \{q+3, \ldots, p-2\}$ such that $D_j \geq q - 2$. For $i = 4, \ldots, p-2$ define $\tilde{m}'_i$ by $(\tilde{m}'_1, \ldots, \tilde{m}'_{p-2}) = (\tilde{m}_1, \ldots, \widehat{\tilde{m}_j}, \ldots, \tilde{m}_{p-1})$. Let U'_i be the filter generated by the first i generators $\tilde{m}'_1, \ldots, \tilde{m}'_i$, and for each $i = 4, \ldots, p-2$ let $D'_i = i - 1 - |\mathrm{Out}(\pi(U'_i)) \setminus \mathrm{Out}(\pi(U'_{i-1}))|$. Note that for $i \in \{1, \ldots, j-1\}$ we have $U_i = U'_i$ and hence $D_i = D'_i$. On the other hand for $i \in \{j+1, \ldots, p-1\}$ we have by lemma 2.7

$$\begin{array}{ll} & D_i \\ = & i - 1 - |\mathrm{Out}(\pi(U_i)) \setminus \mathrm{Out}(\pi(U_{i-1}))| \\ \geq & (i-1) - 1 - |\mathrm{Out}(\pi(U'_{i-1})) \setminus \mathrm{Out}(\pi(U'_{i-2}))| \\ = & D'_{i-1}. \end{array}$$

Hence we now have

$$\begin{array}{ll} & D(p) \\ = & \sum_{i=4}^{p-1} D_i \\ = & \sum_{i=4}^{j-1} D_i + D_j + \sum_{i=j+1}^{p-1} D_i \\ \geq & \sum_{i=4}^{j-1} D'_i + D_j + \sum_{i=j}^{p-2} D'_i \\ = & \sum_{i=4}^{p-2} D'_i + D_j \\ \geq & D(p-1) + (q-2) \end{array}$$

By induction hypothesis we have $D(p-1) \geq (q'-2)(p-1-5q'-15)$ where $q' = \lfloor \frac{p-2}{10} \rfloor$, so the last expression in the above display is $\geq (q'-2)(p-1-5q'-15) + (q-2) = (q-2)(p-5q-15)$ proving our theorem.$\square$

From theorems 1.2 and 2.8 we have the following:

Corollary 2.9 *For $D, x_4 : \boldsymbol{N}_4 \to \boldsymbol{N}_0$, where $x \in \{c, e\}$ we have*

1. $\frac{1}{20} \preceq \frac{D(p)}{p^2} \preceq \frac{1}{8}$ *and*

2. $\frac{3}{8} \preceq \frac{x_4(p)}{p^2} \preceq \frac{9}{20}$.$\square$

REMARK (K. Reuter): Since the order dimension of K_{13} is 5 we have that every graph of order dimension 4 can not contain K_{13} as a subgraph, and is therefore a Turán-12 graph. The maximal number, $t_{12}(p)$, of edges of a Turán-12 graph on p vertices satisfies $\lim_{p\to\infty} t_{12}(p)/p^2 = 11/24$ [2, Lemma

7.1.4]. By corollary 2.9 we have $e_4(p)/p^2 \preceq 9/20 < 11/24$ and hence for large p, we have that the set of graphs on p vertices of order dimension 4 is *strictly* contained in the set of all Turán-12 graphs on p vertices.

Acknowledgements: The author would like to thank Klaus Reuter and Bernd Sturmfels for their valuable help improving the introduction of this paper. Thanks also to Günter M. Ziegler for his encouragement.

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August, 1997