Generators and relations for $W_q(K((S)))$ in characteristic 2

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In this note K is a field of characteristic 2. We let S be an indeterminate over K and let $K((S))$ be the field of formal Laurent series in S over K.

The purpose of the note is to prove the following two theorems about the Witt group $W_q(K((S)))$ of quadratic forms over $K((S))$.

Theorem 1: $W_q(K((S)))$ is the additive group generated by the elements $[\alpha, \beta S^{-k}]$ and $[\alpha S^{-1}, \beta S^{-k+1}]$, where $k \in \mathbf{Z}$, $k \geq 0$, and $\alpha, \beta \in K$, with the condition that $[\alpha, \beta S^{-k}]$ and $[\alpha S^{-1}, \beta S^{-k+1}]$ are biadditive as functions of α and β for each k, and satisfying the following sets of relations:

(I)

$[\alpha, \beta\rho^2 S^{-k}] + [\beta, \alpha\rho^2 S^{-k}] = 0$ if k is even.
$[\alpha S^{-1}, \beta\rho^2 S^{-k+1}] + [\beta S^{-1}, \alpha\rho^2 S^{-k+1}] = 0$ if k is even.
$[\alpha, \beta\rho^2 S^{-k}] + [\beta S^{-1}, \alpha\rho^2 S^{-k+1}] = 0$ if k is odd.

(II)

$[\alpha, \alpha\rho^2 S^{-2k}] + [\alpha, \rho S^{-k}] = 0.$
$[\alpha S^{-1}, \alpha\rho^2 S^{-2k+1}] + [\alpha S^{-1}, \rho S^{-k+1}] = 0.$

Here k runs through the non-negative integers and α, β and ρ run through K.

Theorem 2: For $m \geq 0$ let $W_q(K((S)))_m$ be the subgroup of $W_q(K((S)))$ generated by the $[\alpha, \beta S^{-k}]$ and $[\alpha S^{-1}, \beta S^{-k+1}]$, where $k \in \mathbf{Z}$, $0 \leq k \leq m$ and $\alpha, \beta \in K$. Then:

$W_q(K((S)))_0$ is isomorphic to $W_q(K) \oplus W_q(K)$. A generator $[\alpha, \beta]$ of $W_q(K((S)))_0$ corresponds to $[\alpha, \beta]$ in the first summand $W_q(K)$ but a generator $[\alpha S^{-1}, \beta S]$ corresponds to $[\alpha, \beta]$ in the second summand.

If $n > 0$ then $W_q(K((S)))_{2n}/W_q(K((S)))_{2n-1}$ is isomorphic to $K \wedge_{K^2} K \oplus K \wedge_{K^2} K$. The class of a generator $[\alpha, \beta S^{-2n}]$ corresponds to $\alpha \wedge \beta$ in the first summand but the class of a generator $[\alpha S^{-1}, \beta S^{-2n+1}]$ corresponds to $\alpha \wedge \beta$ in the second summand.

If $n \geq 0$ then $W_q(K((S)))_{2n+1}/W_q(K((S)))_{2n}$ is isomorphic to $K \otimes_{K^2} K$. The class of a generator $[\alpha, \beta S^{-2n-1}]$ corresponds to $\alpha \otimes \beta$ but the class of a generator $[\alpha S^{-1}, \beta S^{-2n}]$ corresponds to $\beta \otimes \alpha$.

We shall also deduce similar results about the Witt group $W_q(K[T])$ of quadratic forms over the ring $K[T]$ of polynomials over K.

Lemma 3: Let $a, b \in K((S))$ such that $ab \in SK[[S]]$. Then $[a, b]$ is hyperbolic.

Proof: Write $ab = Sc$ with $c \in K[[S]]$. By Hensel's Lemma there is an $x \in K[[S]]$ such that $Sx^2 + x = c$. It follows that $(Sx)^2 + (Sx) = Sc = ab$. So the Arf invariant of $[a, b]$ is trivial, and, therefore, $[a, b]$ is hyperbolic.

In particular, if $\alpha, \beta \in K$ and i and j are integers then $[\alpha S^i, \beta S^j] = 0$ in $W_q(K((S)))$ if $i + j > 0$.

Lemma 4: $W_q(K((S)))$ is generated by the $[\alpha S^i, \beta S^j]$, where $\alpha, \beta \in K$ and i and j are integers, $i + j \leq 0$.

Proof: Let $a, b \in K((S))$. Write $a = \sum_{i=-m}^{\infty} a_i S^i$ and $b = \sum_{j=-n}^{\infty} b_j S^j$ with non-negative integers m and n. Let $a' = \sum_{i=-m}^{n} a_i S^i$ and $a'' = \sum_{i=n+1}^{\infty} a_i S^i$. In the same way, let $b' = \sum_{j=-n}^{m} b_j S^j$ and $b'' = \sum_{j=m+1}^{\infty} b_j S^j$. Then $a = a' + a''$ and $b = b' + b''$, hence $[a, b] = [a', b'] + [a', b''] + [a'', b'] + [a'', b'']$ in $W_q(K((S)))$. But $a'b''$, $a''b'$ and $a''b''$ all lie in $SK[[S]]$, so $[a', b'']$, $[a'', b']$ and $[a'', b'']$ are all 0 in $W_q(K((S)))$ by Lemma 3. It follows that $[a, b] = [a', b']$ in $W_q(K((S)))$. We conclude that $[a, b] = \sum_{i=-m}^{n} \sum_{j=-n}^{m} [a_i S^i, b_j S^j]$ in $W_q(K((S)))$.

We shall often write elements $a \in K((S))$ as $a = \sum_i a_i S^i$ with coefficients $a_i \in K$, it being understood that $a_i = 0$ for all i close enough to $-\infty$.

Now let $a, b \in K((S))$ and write $a = \sum_i a_i S^i$ and $b = \sum_j b_j S^j$. With the notation used in the proof of Lemma 4 we then get $[a_i S^i, b_j S^j] = 0$ in $W_q(K((S)))$ if $i < -m$ or $j < -n$ because then $a_i = 0$ or $b_j = 0$, respectively. We already remarked that also $[a_i S^i, b_j S^j] = 0$ in $W_q(K((S)))$ if $i + j > 0$. It follows that in the formally (quadruply) infinite sum $\sum_{i,j} [a_i S^i, b_j S^j]$ there are only finitely many non-zero terms. So the sum is really finite and by the proof of Lemma 4 we have $[a, b] = \sum_{i,j} [a_i S^i, b_j S^j]$. We can, of course, also write this as $[a, b] = \sum_{i,j;i+j\leq 0} [a_i S^i, b_j S^j]$.

For every pair (i, j) of integers the map $K \times K \to W_q(K((S)))$, $(\alpha, \beta) \mapsto [\alpha S^i, \beta S^j]$, is biadditive. It therefore gives rise to an additive map $KS^i \otimes KS^j \to W_q(K((S)))$, $\alpha S^i \otimes \beta S^j \mapsto [\alpha S^i, \beta S^j]$. (The tensor products are over $\mathbf{Z}$.) Combining, we get a group morphism $\bigoplus_{i,j;i+j\leq 0} (KS^i \otimes KS^j) \to W_q(K((S)))$. By Lemma 4 it is an epimorphism. Furthermore, it fits into a commutative diagram

$$\begin{array}{ccc} K((S)) \otimes K((S)) & \to & W_q(K((S))) \\ \downarrow & & || \\ \bigoplus_{i,j;i+j\leq 0} (KS^i \otimes KS^j) & \to & W_q(K((S))) \end{array}$$

Here the upper horizontal morphism is given by $a \otimes b \mapsto [a, b]$ and the left hand vertical morphism is given by $a \otimes b \mapsto \sum_{i,j;i+j\leq 0} (a_i S^i \otimes b_j S^j)$ for $a = \sum_i a_i S^i$ and $b = \sum_j b_j S^j$. Clearly, this vertical morphism is an epimorhism.

The upper horizontal morphism in the diagram is also an epimorphism and we know (cf. [A]) that its kernel is generated by the elements

$a \otimes br^2 + b \otimes ar^2$, where $a, b, r \in K((S))$

$a \otimes ar^2 + a \otimes r$, where $a, r \in K((S))$

in $K((S)) \otimes K((S))$. It follows that the kernel of the lower horizontal morphism in the diagram is generated by the images of these elements under the left vertical morphism.

Let us first look at the images of the elements of the former type. If $a = \sum_i a_i S^i$, $b = \sum_j b_j S^j$ and $r = \sum_q r_q S^q$ then $br^2 = \sum_{j,q} b_j r_q^2 S^{j+2q}$, hence $a \otimes br^2$ maps to $\sum_{i,j,q;i+j+2q\leq 0} a_i S^i \otimes b_j r_q^2 S^{j+2q}$. In the same way, $b \otimes ar^2$ maps to $\sum_{j,i,q;j+i+2q\leq 0} b_j S^j \otimes a_i r_q^2 S^{i+2q}$. So $a \otimes br^2 + b \otimes ar^2$ maps to $\sum_{i,j,q;i+j+2q\leq 0}(a_i S^i \otimes b_j r_q^2 S^{j+2q} + b_j S^j \otimes a_i r_q^2 S^{i+2q})$. We conclude that the image of the subgroup of $K((S)) \otimes K((S))$ generated by elements of the former type is the subgroup of $\bigoplus_{i,j;i+j\leq 0}(KS^i \otimes KS^j)$ generated by the elements

$$\alpha S^i \otimes \beta\rho^2 S^{j+2q} + \beta S^j \otimes \alpha\rho^2 S^{i+2q},$$
where $i, j, q \in Z$, $i + j + 2q \leq 0$, and $\alpha, \beta, \rho \in K$.

Let us now look at the images of the elements of the latter type. Writing a and r as above we get that $a \otimes ar^2$ maps to $\sum_{i,j,q;i+j+2q\leq 0} a_i S^i \otimes a_j r_q^2 S^{j+2q}$ and $a \otimes r$ maps to $\sum_{i,q;i+q\leq 0} a_i S^i \otimes r_q S^q$. If $i < j$ then $a_i S^i \otimes a_j r_q^2 S^{j+2q} + a_j S^j \otimes a_i r_q^2 S^{i+2q}$ lies in the image of the subgroup of $K((S)) \otimes K((S))$ generated by elements of the former type. Module that image we can therefore skip the mixed terms in i and j in the first sum. There remains $\sum_{i,q;i+q\leq 0}(a_i S^i \otimes a_i r_q^2 S^{i+2q} + a_i S^i \otimes r_q S^q)$. We conclude that modulo the mentioned subgroup the image of the subgroup of $K((S)) \otimes K((S))$ generated by elements of the latter type is the subgroup of $\bigoplus_{i,j;i+j\leq 0}(KS^i \otimes KS^j)$ generated by the elements

$$\alpha S^i \otimes \alpha\rho^2 S^{i+2q} + \alpha S^i \otimes \rho S^q,$$
where $i, q \in \mathbf{Z}$, $i + q \leq 0$, and $\alpha, \rho \in K$.

We write our result as a proposition.

Proposition 5: $W_q(K((S)))$ is the additive group generated by the elements $[\alpha S^i, \beta S^j]$, where $i, j \in \mathbf{Z}$ such that $i+j \leq 0$ and $\alpha, \beta \in K$, with the condition that $[\alpha S^i, \beta S^j]$ is biadditive as a function of α and β for each such pair (i, j), and satisfying the following sets of relations:

(I) $[\alpha S^i, \beta\rho^2 S^{j+2q}] + [\beta S^j, \alpha\rho^2 S^{i+2q}] = 0$
for all $i, j, q \in \mathbf{Z}$, $i + j + 2q \leq 0$, and all $\alpha, \beta, \rho \in K$.

(II) $[\alpha S^i, \alpha\rho^2 S^{i+2q}] + [\alpha S^i, \rho S^q] = 0$
for all $i, q \in \mathbf{Z}$, $i + q \leq 0$, and all $\alpha, \rho \in K$.

Letting $\rho = 1$ in relations of type (I) in Proposition 5 we get the relations $[\alpha S^i, \beta S^{j+2q}] + [\beta S^j, \alpha S^{i+2q}] = 0$. Taking $q = 0$ we get $[\alpha S^i, \beta S^j] = [\beta S^j, \alpha S^i]$, because K is a 2-torsion group. So we can rewrite the relations above as $[\alpha S^{i+2q}, \beta S^j] = [\alpha S^i, \beta S^{j+2q}]$. From this we get that

$$[\alpha S^i, \beta S^j] = [\alpha S^0, \beta S^{i+j}] \text{ if } i \text{ is even.}$$

$[\alpha S^i, \beta S^j] = [\alpha S^{-1}, \beta S^{i+j+1}]$ if i is odd.
In particular, $W_q(K((S)))$ is generated by the $[\alpha S^0, \beta S^{-k}]$ and $[\alpha S^{-1}, \beta S^{-k+1}]$, where $k \in \mathbf{Z}$, $k \geq 0$, and $\alpha, \beta \in K$.

Using these formulas in the sets of relations of Proposition 5, we get

(I)

$[\alpha S^0, \beta\rho^2 S^{i+j+2q}] + [\beta S^0, \alpha\rho^2 S^{i+j+2q}] = 0$ if i is even and j is even.
$[\alpha S^0, \beta\rho^2 S^{i+j+2q}] + [\beta S^{-1}, \alpha\rho^2 S^{i+j+2q+1}] = 0$ if i is even and j is odd.
$[\alpha S^{-1}, \beta\rho^2 S^{i+j+2q+1}] + [\beta S^0, \alpha\rho^2 S^{i+j+2q}] = 0$ if i is odd and j is even.
$[\alpha S^{-1}, \beta\rho^2 S^{i+j+2q+1}] + [\beta S^{-1}, \alpha\rho^2 S^{i+j+2q+1}] = 0$ if i is odd and j is odd.

(II)

$[\alpha S^0, \alpha\rho^2 S^{2i+2q}] + [\alpha S^0, \rho S^{i+q}] = 0$ if i is even.
$[\alpha S^{-1}, \alpha\rho^2 S^{2i+2q+1}] + [\alpha S^{-1}, \rho S^{i+q+1}] = 0$ if i is odd.

Renaming the exponents, this becomes

(I)

$[\alpha, \beta\rho^2 S^{-k}] + [\beta, \alpha\rho^2 S^{-k}] = 0$ if k is even.
$[\alpha, \beta\rho^2 S^{-k}] + [\beta S^{-1}, \alpha\rho^2 S^{-k+1}] = 0$ if k is odd.
$[\alpha S^{-1}, \beta\rho^2 S^{-k+1}] + [\beta, \alpha\rho^2 S^{-k}] = 0$ if k is odd.
$[\alpha S^{-1}, \beta\rho^2 S^{-k+1}] + [\beta S^{-1}, \alpha\rho^2 S^{-k+1}] = 0$ if k is even.

(II)

$[\alpha, \alpha\rho^2 S^{-2k}] + [\alpha, \rho S^{-k}] = 0.$
$[\alpha S^{-1}, \alpha\rho^2 S^{-2k+1}] + [\alpha S^{-1}, \rho S^{-k+1}] = 0.$

Here, k is an integer, $k \geq 0$, and $\alpha, \beta \in K$. We also have written simply α instead of αS^0.

Noting that the second and third sets of relations of type (I) are the same, except that α and β are interchanged, we now have proven Theorem 1.

The presentation may, however, been somewhat lax in this last part. What really has been shown is that the morphism $\bigoplus_{i,j;i+j\leq 0}(KS^i \otimes KS^j) \to W_q(K((S)))$ factors over $\bigoplus_{k;k\geq 0}((KS^0 \otimes KS^{-k}) \oplus (KS^{-1} \otimes KS^{-k+1}))$ by mapping $\alpha S^i \otimes \beta S^j$ to $\alpha S^0 \otimes \beta S^{i+j}$ if i is even but to $\alpha S^{-1} \otimes \beta S^{i+j+1}$ if i is odd. The new relations are the images of the old ones under this morphism.

To simplify the presentation we now write $T = S^{-1}$. The generators of $W_q(K((S)))$, according to Theorem 1, then are the

$[\alpha, \beta T^k]$ and $[\alpha T, \beta T^{k-1}]$, where $k \in \mathbf{Z}$, $k \geq 0$, and $\alpha, \beta \in K$.

and the relations are:

(I)

$[\alpha, \beta\rho^2 T^k] + [\beta, \alpha\rho^2 T^k] = 0$, k even.
$[\alpha T, \beta\rho^2 T^{k-1}] + [\beta T, \alpha\rho^2 T^{k-1}] = 0$, k even.

$[\alpha, \beta\rho^2 T^k] + [\beta T, \alpha\rho^2 T^{k-1}] = 0$, k odd.

(II)

$[\alpha, \alpha\rho^2 T^{2k}] + [\alpha, \rho T^k] = 0.$
$[\alpha T, \alpha\rho^2 T^{2k-1}] + [\alpha T, \rho T^{k-1}] = 0.$

We call k the degree of the generators $[\alpha, \beta T^k]$ and $[\alpha T, \beta T^{k-1}]$. (More precisely, it is the degree of the corresponding elements in $K \otimes KT^k$ and $KT \otimes KT^{k-1}$, respectively.) We also refer to generators $[\alpha, \beta T^k]$ as even and generators $[\alpha T, \beta T^{k-1}]$ as odd. We sometimes refer to the relations above as basic. A general relation is a **Z**-linear combination of the basic ones.

By symmetry, $[\alpha, \beta T^k] = [\beta T^k, \alpha]$ and $[\alpha T, \beta T^{k-1}] = [\beta T^{k-1}, \alpha T]$. It follows that $[\alpha, \beta T^k] = [\beta, \alpha T^k]$ and $[\alpha T, \beta T^{k-1}] = [\beta T, \alpha T^{k-1}]$ if k is even but $[\alpha, \beta T^k] = [\beta T, \alpha T^{k-1}]$ and $[\alpha T, \beta T^{k-1}] = [\beta, \alpha T^k]$ if k is odd.

For $m \geq 0$ we let $W_q(K((S)))_m$ be the subgroup of $W_q(K((S)))$ generated by generators of degree $\leq m$.

The generators of degree 0 are the $[\alpha, \beta]$ and $[\alpha T, \beta T^{-1}]$, $\alpha, \beta \in K$, and the only (basic) relations in which they occur are

(I)

$[\alpha, \beta\rho^2] + [\beta, \alpha\rho^2] = 0.$
$[\alpha T, \beta\rho^2 T^{-1}] + [\beta T, \alpha\rho^2 T^{-1}] = 0.$

(II)

$[\alpha, \alpha\rho^2] + [\alpha, \rho] = 0.$
$[\alpha T, \alpha\rho^2 T^{-1}] + [\alpha T, \rho T^{-1}] = 0.$

It follows that $W_q(K((S)))_0$ is the direct sum of two copies of $W_q(K)$, an even one and an odd one. The even one is the natural image of $W_q(K)$ in $W_q(K((S)))$.

We now want to compute the quotients $W_q(K((S)))_m / W_q(K((S)))_{m-1}$ for $m > 0$. We shall, in fact, determine the relations between the generators of $W_q(K((S)))_m$. It turns out that they are generated by the basic relations of degree $\leq m$. This is, however, not trivial because a general relation between these generators, being a sum of our basic relations, might contain summands where generators of degree $> m$ occur.

Let B be the additive group with generators $\lfloor \alpha, \beta T^k \rfloor$ and $\lfloor \alpha T, \beta T^{k-1} \rfloor$, where $k \geq 0$ and $\alpha, \beta \in K$, with the condition that $\lfloor \alpha, \beta\rho^2 T^k \rfloor$ and $\lfloor \alpha T, \beta T^{k-1} \rfloor$ are biadditive as functions of α and β for each k, and satisfying the following sets of relations:

$\lfloor \alpha, \beta\rho^2 T^k \rfloor + \lfloor \beta, \alpha\rho^2 T^k \rfloor = 0$ if k is even.
$\lfloor \alpha T, \beta\rho^2 T^{k-1} \rfloor + \lfloor \beta T, \alpha\rho^2 T^{k-1} \rfloor = 0$ if k is even.

$\lfloor \alpha, \beta\rho^2 T^k \rfloor + \lfloor \beta T, \alpha\rho^2 T^{k-1} \rfloor = 0$ if k is odd.

Here k runs through the non-negative integers and α, β and ρ run through K.

As the relations are homogeneous, B is a graded group. We denote by B^m the subgroup of homogeneous elements of degree m. We want to describe each B^m.

Assume first that $m = 2n$ is even. The relations in B^{2n} are:

$\lfloor \alpha, \beta\rho^2 T^{2n} \rfloor + \lfloor \beta, \alpha\rho^2 T^{2n} \rfloor = 0.$
$\lfloor \alpha T, \beta\rho^2 T^{2n-1} \rfloor + \lfloor \beta T, \alpha\rho^2 T^{2n-1} \rfloor = 0.$

Here α, β and ρ run through K. So there are no relations connecting the even and the odd generateors. It follows that B^{2n} is the direct sum of the even part and the odd part. Each part is isomorphic to the group with generators $\lfloor \alpha, \beta \rfloor^-$, where $\alpha, \beta \in K$, with the condition that $\lfloor \alpha, \beta \rfloor^-$ is biadditive as a function of α and β, and satisfying the following set of relations:

$\lfloor \alpha, \beta\rho^2 \rfloor^- + \lfloor \beta, \alpha\rho^2 \rfloor^- = 0.$

Here α, β and ρ run through K. Taking $\rho = 1$ in the relations we get $\lfloor \alpha, \beta \rfloor^- + \lfloor \beta, \alpha \rfloor^- = 0$, i.e., $\lfloor \alpha, \beta \rfloor^- = \lfloor \beta, \alpha \rfloor^-$. Using that to rewrite the relations, they say that $\lfloor \alpha, \beta\rho^2 \rfloor^- = \lfloor \alpha\rho^2, \beta \rfloor^-$. It follows that this group is canonically isomorphic to the symmetric tensor product $K \hat{\otimes}_{K_0} K$, where K_0 is the subfield K^2 of K.

Now assume that $m = 2n + 1$ is odd. The relations in B^{2n+1} are:

$\lfloor \alpha, \beta\rho^2 T^{2n+1} \rfloor + \lfloor \beta T, \alpha\rho^2 T^{2n} \rfloor = 0.$

Here α, β and ρ run through K. Taking $\rho = 1$ in these relations, we get $\lfloor \beta T, \alpha T^{2n} \rfloor = \lfloor \alpha, \beta T^{2n+1} \rfloor$. It therefore suffice to use the even generators $\lfloor \alpha, \beta T^{2n+1} \rfloor$. The relations now become:

$\lfloor \alpha, \beta\rho^2 T^{2n+1} \rfloor + \lfloor \alpha\rho^2, \beta T^{2n+1} \rfloor = 0.$

Writing $\lfloor \alpha, \beta \rfloor^-$ instead of $\lfloor \alpha, \beta T^{2n+1} \rfloor$, the relations now become $\lfloor \alpha, \beta\rho^2 \rfloor^- + \lfloor \alpha\rho^2, \beta \rfloor^- = 0$, i.e., $\lfloor \alpha, \beta\rho^2 \rfloor^- = \lfloor \alpha\rho^2, \beta \rfloor^-$. It follows that this group is canonically isomorphic to the tensor product $K \otimes_{K_0} K$. (Note that this is not the symmetric tensor product.)

We have an obvious epimorphism $B \to W_q(K((S)))$. Its kernel is generated by the

$\lfloor \alpha, \alpha\rho^2 T^{2k} \rfloor + \lfloor \alpha, \rho T^k \rfloor.$
$\lfloor \alpha T, \alpha\rho^2 T^{2k-1} \rfloor + \lfloor \alpha T, \rho T^{k-1} \rfloor.$

Lemma 6: The symbols $K \times K \to B^{2n}$,

$$(\alpha, \rho) \mapsto \lfloor \alpha, \alpha\rho^2 T^{2n} \rfloor \quad \text{and} \quad (\alpha, \rho) \mapsto \lfloor \alpha T, \alpha\rho^2 T^{2n-1} \rfloor$$

are biadditive.

Proof: Additivity in ρ is clear. There remains to prove the additivity in α. As the proof is the same for both symbols, we do the computations only for the latter one. We have

$$\begin{aligned} & \lfloor(\alpha+\beta)T,(\alpha+\beta)\rho^2T^{2n-1}\rfloor \\ = & \lfloor\alpha T,\alpha\rho^2T^{2n-1}\rfloor+\lfloor\alpha T,\beta\rho^2T^{2n-1}\rfloor+\lfloor\beta T,\alpha\rho^2T^{2n-1}\rfloor+\lfloor\beta T,\beta\rho^2T^{2n-1}\rfloor \\ = & \lfloor\alpha T,\alpha\rho^2T^{2n-1}\rfloor+\lfloor\beta T,\beta\rho^2T^{2n-1}\rfloor \end{aligned}$$

because $\lfloor\alpha T,\beta\rho^2T^{2n-1}\rfloor+\lfloor\beta T,\alpha\rho^2T^{2n-1}\rfloor=0$ in B^{2n}.

Lemma 7: Let $n>0$. Assume that the homogeneous component of degree $2n$ of a sum

$$\sum_i\Big(\lfloor\alpha_i,\alpha_i\rho_i^2T^{2n}\rfloor+\lfloor\alpha_i,\rho_iT^n\rfloor\Big)+\sum_j\Big(\lfloor\beta_jT,\beta_j\sigma_j^2T^{2n-1}\rfloor+\lfloor\beta_jT,\sigma_jT^{n-1}\rfloor\Big)$$

in B is trivial. Then the whole sum is trivial.

Proof: As B^{2n} is the direct sum of the even part and the odd part, we can look at each part separately. As the proof is the same for both parts, we only consider the even part.

So we have $\sum_i\lfloor\alpha_i,\alpha_i\rho_i^2T^{2n}\rfloor=0$ in B^{2n}. We choose a basis $(\omega_\mu)_\mu$ for K over K_0 and write each α_i as $\alpha_i=\sum_\mu\omega_\mu\alpha_{i,\mu}^2$ with $\alpha_{i,\mu}\in K$. By additivity, we then have

$$\begin{aligned} & \sum_i\lfloor\alpha_i,\alpha_i\rho_i^2T^{2n}\rfloor \\ = & \sum_i\sum_\mu\lfloor\omega_\mu\alpha_{i,\mu}^2,\omega_\mu\alpha_{i,\mu}^2\rho_i^2T^{2n}\rfloor=\sum_\mu\sum_i\lfloor\omega_\mu\alpha_{i,\mu}^2,\omega_\mu\alpha_{i,\mu}^2\rho_i^2T^{2n}\rfloor \\ = & \sum_\mu\sum_i\lfloor\omega_\mu,\omega_\mu\alpha_{i,\mu}^4\rho_i^2T^{2n}\rfloor=\sum_\mu\lfloor\omega_\mu,\omega_\mu\sum_i\alpha_{i,\mu}^4\rho_i^2T^{2n}\rfloor \end{aligned}$$

This corresponds to $\sum_\mu\Big((\omega_\mu\hat{\otimes}_{K_0}\omega_\mu)\sum_i\alpha_{i,\mu}^4\rho_i^2\Big)$ in $K\hat{\otimes}_{K_0}K$. Hence this being 0 means that for each μ the sum $\sum_i\alpha_{i,\mu}^4\rho_i^2=0$, i.e., the sum $\sum_i\alpha_{i,\mu}^2\rho_i=0$.

We now get

$$\begin{aligned} & \sum_i\lfloor\alpha_i,\rho_iT^n\rfloor \\ = & \sum_i\sum_\mu\lfloor\omega_\mu\alpha_{i,\mu}^2,\rho_iT^n\rfloor=\sum_\mu\sum_i\lfloor\omega_\mu\alpha_{i,\mu}^2,\rho_iT^n\rfloor \\ = & \sum_\mu\sum_i\lfloor\omega_\mu,\alpha_{i,\mu}^2\rho_iT^n\rfloor=\sum_\mu\lfloor\omega_\mu,\sum_i\alpha_{i,\mu}^2\rho_iT^n\rfloor=0 \end{aligned}$$

because each sum $\sum_i \alpha_{i,\mu}^2 \rho_i = 0$.

Now let B_m be the subgroup of B of elements of degree $\leq m$. We have an obvious epimorphism $B_m \to W_q(K((S)))_m$. From Lemma 7 it now easily follows that its kernel is generated by the $\lfloor \alpha, \alpha\rho^2 T^{2k} \rfloor + \lfloor \alpha, \rho T^k \rfloor$ and the $\lfloor \alpha T, \alpha\rho^2 T^{2k-1} \rfloor + \lfloor \alpha T, \rho T^{k-1} \rfloor$, where $k = 0, 1, \ldots, m$ and α and ρ run through K.

From this it follows that the relations between the generators of $W_q(K((S)))_m$ are generated by the basic relations of degree $\leq m$. This makes it easy to compute the quotient $W_q(K((S)))_m / W_q(K((S)))_{m-1}$ for $m > 0$. If m is odd then this is isomorphic to B^m and we already noted that then B^m is isomorphic to $K \otimes_{K_0} K$. But if $m = 2n$ is even then we get that the kernel of the canonical epimorphism $B^{2n} \to W_q(K((S)))_{2n} / W_q(K((S)))_{2n-1}$ is generated by the $\lfloor \alpha, \alpha\rho^2 T^{2n} \rfloor$ and the $\lfloor \alpha T, \alpha\rho^2 T^{2n-1} \rfloor$, where α and ρ run through K. In view of our earlier description of B^{2n} as the direct sum of two copies of $K \hat{\otimes}_{K_0} K$, it follows that $W_q(K((S)))_{2n} / W_q(K((S)))_{2n-1}$ is isomorphic to the direct sum of two copies of the second exterior product $K \wedge_{K_0} K$.

Considering the definitions of the isomorphisms involved, we now have proven Theorem 2.

It is natural to ask what consequences our results have for the quotients $I^d W_q(K((S))) / I^{d+1} W_q(K((S)))$. Here, we only have a brief look at the cases $d = 0$ and $d = 1$.

We know that $IW_q(K((S)))$ is generated by the $[[a, b]] := [a, b] + [1, ab]$ with $a, b \in K((S))$. Going through our steps, we first see that it is generated by the $[\alpha S^i, \beta S^j] + [1, \alpha\beta S^{i+j}]$ with $i, j \in \mathbf{Z}$ such that $i + j \leq 0$ and $\alpha, \beta \in K$, and then that it is generated by the $[\alpha, \beta T^k] + [1, \alpha\beta T^k]$ and $[\alpha T, \beta T^{k-1}] + [1, \alpha\beta T^k]$ with $k \in \mathbf{Z}$ such that $k \geq 0$ and $\alpha, \beta \in K$.

Our (ascending) filtration of $W_q(K((S)))$ induces a filtration of the quotient $W_q(K((S))) / IW_q(K((S)))$. As the additional relations are homogenous, it is easy to compute the quotients of this filtration from our results on the filtration of $W_q(K((S)))$.

In degree 0 we get a copy of $W_q(K) / IW_q(K)$. In even degree $2n > 0$ we get a copy of $K \wedge_{K_0} K$ modulo the subgroup generated by the $\alpha \wedge \beta + 1 \wedge \alpha\beta$, $\alpha, \beta \in K$. By $\alpha \wedge \beta \mapsto \alpha\beta + K_0$, this is easily seen to be isomorpic to K/K_0. In odd degree $2n+1$ we get a copy of $K \otimes_{K_0} K$ modulo the subgroup generated by the $\alpha \otimes \beta + 1 \otimes \alpha\beta$ and $\beta \otimes \alpha + 1 \otimes \alpha\beta$, $\alpha, \beta \in K$. By $\alpha \otimes \beta \mapsto \alpha\beta$, this is easily seen to be isomorpic to K.

(We know that $[1, K((S))]$ is a complement to $IW_q(K((S)))$ in $W_q(K((S)))$. So we could study $[1, K((S))]$ instead of $W_q(K((S))) / IW_q(K((S)))$. But we would have to show that the induced filtations agree.)

Now $W_q(K((S)))/IW_q(K((S)))$ is isomorphic to $K((S))/\wp(K((S)))$ and it is easy to get the filtration and the description of the quotients by working directly with $K((S))/\wp(K((S)))$. But we think of our presentation as an introduction to the next step.

We know that $IW_q(K((S)))$ is isomorphic to $W_q(K((S)))/[1,(K((S)))]$, $[[a,b]]$ corresponding to the class of $[a,b]$. Also, $I^2W_q(K((S)))$ is generated by the $[[a,bc]]+[[b,ac]]+[[ab,c]]$, $a,b,c \in K((S))$. We therefore get a description of the quotient $IW_q(K((S)))/I^2W_q(K((S)))$ as the quotient of $W_q(K((S)))$ by the subgroup generated by the $[1,a]$, $a \in K((S))$ and the $[[a,bc]]+[[b,ac]]+[[ab,c]]$, $a,b,c \in K((S))$.

Going through our our steps, we see that it suffices to use the generators $[1,\alpha T^k]$, $k \geq 0$ and $\alpha \in K$, and

$[\alpha,\beta\gamma T^k]+[\beta,\alpha\gamma T^k]+[\alpha\beta,\gamma T^k]$
$[\alpha,\beta\gamma T^k]+[\beta T,\alpha\gamma T^{k-1}]+[\alpha\beta T,\gamma T^{k-1}]$
$[\alpha T,\beta\gamma T^{k-1}]+[\beta,\alpha\gamma T^k]+[\alpha\beta T,\gamma T^{k-1}]$
$[\alpha T,\beta\gamma T^{k-1}]+[\beta T,\alpha\gamma T^{k-1}]+[\alpha\beta,\gamma T^k]$

with $k \geq 0$ and $\alpha,\beta,\gamma \in K$. Of the last four types of generators the third is simply the second with α and β interchanged. So the third is not needed. Interchanging α and γ in the second and rearranging we get $[\beta\gamma T,\alpha T^{k-1}]+[\beta T,\alpha\gamma T^{k-1}]+[\gamma,\alpha\beta T^k]$. Using a remark on the symmetry of our generators, we see that this is of the fourth type if k is even. Interchanging β and γ in the second and rearranging we get $[\alpha,\beta\gamma T^k]+[\alpha\gamma T,\beta T^{k-1}]+[\gamma T,\alpha\beta T^{k-1}]$. Using a remark on the symmetry of our generators, we see that this is of the first type if k is odd. So the second type is also not needed. Finally, letting $\beta=\gamma=1$ in the first type, we get $[1,\alpha T^k]$. So it suffices to use the generators

$[\alpha,\beta\gamma T^k]+[\beta,\alpha\gamma T^k]+[\alpha\beta,\gamma T^k]$
$[\alpha T,\beta\gamma T^{k-1}]+[\beta T,\alpha\gamma T^{k-1}]+[\alpha\beta,\gamma T^k]$

with $k \geq 0$ and $\alpha,\beta,\gamma \in K$.

Letting $\alpha = 1$ in generators of the last type, we get $[T,\beta\gamma T^{k-1}]+[\beta T,\gamma T^{k-1}]+[\beta,\gamma T^k]$. It follows that modulo our subgroup we have $[\beta,\gamma T^k] \equiv [T,\beta\gamma T^{k-1}]+[\beta T,\gamma T^{k-1}]$. Using this to eliminate the even generators from both remaining types of generators, we get $[T,\alpha\beta\gamma T^{k-1}]+[\alpha T,\beta\gamma T^{k-1}]+[\beta T,\alpha\gamma T^{k-1}]+[\alpha\beta T,\gamma T^{k-1}]$ in both cases. This means that it also suffices to use the generators

$[T,\alpha\beta T^{k-1}]+[\alpha T,\beta T^{k-1}]+[\alpha,\beta T^k]$
$[T,\alpha\beta\gamma T^{k-1}]+[\alpha T,\beta\gamma T^{k-1}]+[\beta T,\alpha\gamma T^{k-1}]+[\alpha\beta T,\gamma T^{k-1}]$

with $k \geq 0$ and $\alpha,\beta,\gamma \in K$.

From the above we get a filtration of $IW_q(K((S)))/I^2W_q(K((S)))$. As our generators for our subgroup are homogenous, it is easy to get a representation for the quotients for this filtration.

In degree 0 we get a copy of $W_q(K)$ (the second one) modulo the subgroup generated by the $[1, \alpha\beta\gamma] + [\alpha, \beta\gamma] + [\beta, \alpha\gamma] + [\alpha\beta, \gamma]$. But this subgroup is $I^2 W_q(K)$, so the degree 0 part is isomorphic to $W_q(K)/I^2 W_q(K)$. This representation is not quite satisfactory because the constant part is not clear. It is better to use the generators $[T, \alpha\beta T^{-1}] + [\alpha T, \beta T^{-1}] + [\alpha, \beta]$ to rewrite this as $IW_q(K)/I^2 W_q(K) \oplus K/\wp(K)$. Here the first summand is the constant part — the part coming from K — and the second part corresponds to the generators $[T, \alpha T^{-1}]$, $a \in K$. Looking at $\mathrm{Br}_2(K((S)))$ instead of $IW_q(K((S)))/I^2 W_q(K((S)))$ this means that the degree 0 part of $\mathrm{Br}_2(K((S)))$ is $\mathrm{Br}_2(K)$ plus the classes of the quaternion algebras $((T, \alpha T^{-1}))_{K((S))}$, $a \in K$.

In even degree $2n > 0$ we get a copy of $K \wedge_{K_0} K$ (the second one) modulo the subgroup generated by the $1 \wedge \alpha\beta\gamma + \alpha \wedge \beta\gamma + \beta \wedge \alpha\gamma + \alpha\beta \wedge \gamma$. We have not found a simple description of this quotient.

In odd degree $2n + 1$ we get a copy of $K \otimes_{K_0} K$ modulo the subgroup generated by the $\alpha\beta \otimes 1 + \beta \otimes \alpha + \alpha \otimes \beta$ and the $\alpha\beta\gamma \otimes 1 + \beta\gamma \otimes \alpha + \alpha\gamma \otimes \beta + \gamma \otimes \alpha\beta$. Using the former type of generators to change the order of the tensor products in the latter, we can replace the latter by the $\alpha \otimes \beta\gamma + \beta \otimes \alpha\gamma + \alpha\beta \otimes \gamma$. Letting $\gamma = 1$ in this we get the former type. It follows that the generators of the type $\alpha \otimes \beta\gamma + \beta \otimes \alpha\gamma + \alpha\beta \otimes \gamma$ suffice. Looking at $K \otimes_{K_0} K$ as a right vector space over K, we now see that our quotient is the quotient space of $K \otimes_{K_0} K$ by the subspace generated by the $\alpha \otimes \beta + \beta \otimes \alpha + \alpha\beta \otimes 1$. But this quotient is isomorphic to the space of differentials of K, the class of $\alpha \otimes \beta$ corresponding to $\beta \, d\alpha$.

For every pair (i, j) of integers the map $K \times K \to W_q(K(S))$, $(\alpha, \beta) \mapsto [\alpha S^i, \beta S^j]$, is biadditive. Also, the relations in Propositon 5 hold already in $W_q(K(S))$. It follows that we have a group morphism $W_q(K((S))) \to W_q(K(S))$ mapping the generator $[\alpha S^i, \beta S^j]$ of $W_q(K((S)))$ to $[\alpha S^i, \beta S^j]$ in $W_q(K(S))$. Clearly, the composition of this morphism $W_q(K((S))) \to W_q(K(S))$ and the morphism $W_q(K(S)) \to W_q(K((S)))$ induced by the inclusion $K(S) \subseteq K((S))$ is the identity on $W_q(K((S)))$. (In fact, we can do the same for $W_q(K[S, S^{-1}])$ instead of $W_q(K(S))$.)

Let us write $T = S^{-1}$ like we did just after finishing the proof of Theorem 1 and consider the description given there of the generators and relations for $W_q(K((S)))$ in terms of T.

The only generators having a negative exponent on T are the $[\alpha T, \beta T^{-1}]$. The only (basic) relations in which they occur are the

(I) $[\alpha T, \alpha\rho^2 T^{-1}] + [\alpha T, \rho T^{-1}] = 0$, i.e., $[\alpha T, (\alpha\rho^2 + \rho)T^{-1}] = 0$.
(II) $[\alpha T, \beta\rho^2 T^{-1}] + [\beta T, \alpha\rho^2 T^{-1}] = 0$.

It follows that the subgroup $W_q(K((S)))''$ of $W_q(K((S)))$ generated by these generators is isomorphic to $W_q(K)$. Clearly, $W_q(K((S)))$ is the direct sum of $W_q(K((S)))'$ and $W_q(K((S)))''$, where $W_q(K((S)))'$ is generated by the remaining generators. Furthermore, the relations between these generators of $W_q(K((S)))'$ are generated by the remaining basic relations in the description.

We already saw that our (basic) relations hold in $W_q(K(S)) = W_q(K(T))$, giving us a monomorphism $W_q(K((S))) \to W_q(K(T))$. The image of $W_q(K((S)))'$ under this monomorphism lies in the image of $W_q(K[T])$ in $W_q(K(T))$. As $W_q(K[T]) \to W_q(K(T))$ is a monomorphism, it follows that this induces a monomorphism $W_q(K((S)))' \to W_q(K[T])$. As $K[T]$ is a principal ideal domain, $W_q(K[T])$ is additively generated by the (classes of the) binary forms $[f, g]$, $f, g \in K[T]$. It follows that it is generated by the $[\alpha T^i, \beta T^j]$, with $i, j \in \mathbf{Z}$, $i, j \geq 0$, and $\alpha, \beta \in K$. But $[\alpha T^{i+2q}, \beta T^j] = [\alpha T^i, \beta T^{j+2q}]$ holds in $W_q(K(T))$, hence also in $W_q(K[T])$. So $W_q(K[T])$ is additively generated by the $[\alpha, \beta T^k]$ and the $[\alpha T, \beta T^k]$, $k \geq 0$. So our morphism $W_q(K((S)))' \to W_q(K[T])$ is also an epimorphism.

All this means that the natural morphism $W_q(K[T]) \to W_q(K((S)))$, induced by the inclusion $K[T] = K[S^{-1}] \subseteq K((S))$, is a monomorphism and that its image is $W_q(K((S)))'$. We get in particular the following theorem.

Theorem 8: $W_q(K[T])$ is the additive group generated by the elements $[\alpha, \beta T^k]$ and $[\alpha T, \beta T^k]$, where $k \in \mathbf{Z}$, $k \geq 0$, and $\alpha, \beta \in K$, with the condition that $[\alpha, \beta T^k]$ and $[\alpha T, \beta T^k]$ are biadditive as functions of α and β for each such pair k, and satisfying the following sets of relations:

(I)

$[\alpha, \beta\rho^2 T^k] + [\beta, \alpha\rho^2 T^k] = 0$, k even.
$[\alpha T, \beta\rho^2 T^{k-1}] + [\beta T, \alpha\rho^2 T^{k-1}] = 0$, k even.
$[\alpha, \beta\rho^2 T^k] + [\beta T, \alpha\rho^2 T^{k-1}] = 0$, k odd.

(II)

$[\alpha, \alpha\rho^2 T^{2k}] + [\alpha, \rho T^k] = 0$.
$[\alpha T, \alpha\rho^2 T^{2k+1}] + [\alpha T, \rho T^k] = 0$.

We also get the following theorem.

Theorem 9: There is a split short exact sequence

$$0 \to W_q(K[T]) \to W_q(K((S))) \to W_q(K) \to 0$$

where $S = T^{-1}$. A splitting $W_q(K) \to W_q(K((S)))$ is given by $[\alpha, \beta] \mapsto [\alpha S^{-1}, \beta S] = \langle S \rangle [\alpha, \beta]$.

Remark: In the case $\mathrm{char}(K) \neq 2$ we have $W_q(K[T]) = W_q(K)$ and the short exact sequence is well known.

From Theorem 2 we get a corresponding filtration of $W_q(K[T])$. The only difference is that in degree 0 the second summand $W_q(K)$ is missing.

Note: We can also use our computations to show that

$$W_q(K[S, S^{-1}]) = (W_q(K[S]) + W_q(K[S^{-1}])) \oplus \langle S \rangle W_q(K)$$

and that

$$W_q(K[S]) \cap W_q(K[S^{-1}]) = W_q(K)$$

(Here we are identifying $W_q(K)$, $W_q(K[S])$ and $W_q(K[S^{-1}])$ with their images in $W_q(K[S, S^{-1}])$).

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