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Abstract

The problem $-\varepsilon^2\Delta u+F(V(x),u)=0$ is studied in all of $\mathbb{R}^n$. For small $\varepsilon>0$, solutions are constructed which concentrate on a prescribed set, possibly infinite, of non-degenerate critical points of the potential function V. When this set is infinite these are so-called infinite-bump solutions. The construction does not require V to be periodic. The solution is exponentially close to the infinite sum of the single bump solutions for small ε. The latter is shown to be non-degenerate.

1 Introduction

This is the first of two papers devoted to the construction of infinite-bump solutions of elliptic semi-linear partial differential equations. We study the problem $-\varepsilon^2\Delta u+F(V(x),u)=0$ in $\mathbb{R}^n$ for small $\varepsilon>0$; in the language of physics, which interprets ε essentially as Planck's constant, this yields so-called semi-classical solutions.

It is convenient to rescale the variable to obtain

$$-\Delta u+F(V(\varepsilon x),u)=0. \tag{1}$$

If we replace x by $x+b/\epsilon$ and let $\epsilon\to 0$, then, disregarding rigour, we obtain the equation

$$-\Delta u+F(a,u)=0, \tag{2}$$

where $a=V(b)$ is a constant. The solutions of (2) play a crucial role in this paper.

A model case is the non-linear Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u - u^p = 0.$$

Floer and Weinstein showed in an influential paper [4] that if $1 < p < (n+2)/(n-2)$ (with no restriction on p in dimensions 1 and 2), and if V satisfies $V(x) > h > 0$, then the non-linear Schrödinger equation in one dimension possesses so-called single-bump solutions for sufficiently small $\varepsilon > 0$. For any given non-degenerate critical point b of V such a solution exists that concentrates at b as $\varepsilon \to 0$. By this is meant that in the complement of any neighbourhood of b the solution converges uniformly to 0 while the maximum of the solution in the neighbourhood remains approximately constant. This definition can be extended to define concentration at a discrete multipoint set B in an obvious way.

These results were extended by Oh in [11] to higher dimensions. Oh then proved in [12] the existence of so-called multi-bump solutions, that concentrate at a given finite set of non-degenerate critical points of V. Multi-bump solutions were then found for various classes of equations using a variety of methods, some variational as in [3] or [5], and some non-variational as in [1] and [7]. In many instances the non-degeneracy assumption on the critical points was relaxed or dropped.

In her Ph.D dissertation [18], Thandi constructed multi-bump solutions that concentrate on an infinite set B of non-degenerate critical points of V. She assumed the potential to be a periodic function so that the set B is finite modulo the period lattice of V. The methods used are an extension of those of Oh: a solution with an arbitrary but finite number k of bumps is constructed, along with bounds which do not depend on k, which permit the taking of the limit as $k \to \infty$.

In this paper we introduce a more direct method, allowing the construction of infinite-bump solutions without first constructing solutions with finitely many bumps. Moreover our methods work for non-periodic potentials V. To our knowledge, this is the first time that infinite-bump solutions have been constructed for this problem without any periodicity assumption. The infinite sum of the single bump solutions is found to approximate the infinite-bump solution with error of order $e^{-\sigma/\varepsilon}$.

Our construction is a slight modification of the ideas introduced in the second author's Ph.D thesis [10]. In [10] the periodicity assumption is retained but is largely used to guarantee an infinite set of single bump solutions that have uniform decay and a uniform asymptotic structure. Our approach now is deliberately "axiomatic", that is, we take for granted in the present paper the existence of infinitely many single-bump solutions to (1) with these uniformity properties and avoid the assumption of periodicity. In this way we can isolate the core of the argument.

This leaves temporarily open the problem of how to obtain examples for non-periodic V that exhibit an infinite set of single bump solutions with the required uniformity properties. This will be treated in a second paper by the authors.

The most important assumption is the existence of a non-trivial, radially symmetric and exponentially decaying solution ϕ_a to the limit equation (2) for a range I of values of a. This solution, the ground state, must satisfy a so-called quasi-non-degeneracy condition, usually expressed by requiring that the operator

$$-\Delta + \frac{\partial F}{\partial u}(a, \phi_a) : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

has as kernel the space spanned by the partial derivatives of ϕ_a, and its range is the subspace of L^2 that is orthogonal to these partial derivatives. In the present case we must replace H^2 and L^2 by the Hölder spaces $C^{2,\lambda}$ and C^λ, since the infinite-bump solutions that we aim for cannot belong to L^2. The ground state is difficult to obtain, but its existence is known in the case of the non-linear Schrödinger equation, see [19] and [6], as well as in many one-dimensional problems.

We give ourselves a (possibly infinite) set B consisting of non-degenerate critical points of V. Corresponding to each b in B, there exists a single-bump solution u_ε^b concentrating at b. Under suitable uniformity and decay conditions on the collection u_ε^b (to be justified in the sequel to this paper) we can construct their sum $\sum_{b\in B} u_\varepsilon^b$, which converges for every small ε. It is then shown that the function represented by this series is non-degenerate. In the context of this paper, this means that the Fréchet derivative of the non-linear operator $g_\varepsilon(u) := -\Delta u + F(V(\varepsilon x), u)$ is invertible at the sum function; more precisely, the operator

$$-\Delta + \frac{\partial F}{\partial u}\Big(V(\varepsilon x), \sum_{b\in B} u_\varepsilon^b\Big) : C^{2,\lambda}(\mathbb{R}^n) \to C^\lambda(\mathbb{R}^n)$$

is invertible for sufficiently small $\varepsilon > 0$ and some choice of the Hölder exponent $\lambda < 1$. Moreover, the norm of its inverse does not blow up faster than ε^{-2} as ε tends to 0. Proving this is distinctly non-trivial and is the first of the two main results of this paper. The second main result follows from this and is based on using the Newton-Kantorovich method to obtain a solution which is asymptotically close to $\sum_b u_\varepsilon^b$ as ε goes to 0. This will be an infinite-bump solution which concentrates precisely at the set B.

Our results can be seen as an asymptotic superposition principle for a non-linear equation, where solutions are glued together to form a new one, provided ε is sufficiently small. This can be compared to the shadowing lemma of Angenent, see [2]. Angenent studied a semi-linear elliptic equation together with a covering of $\mathbb{R}^n$ by open sets Ω^i. Given a solution u^i on each Ω^i he found a solution U on $\mathbb{R}^n$ whose restriction to each Ω^i is near u^i. The result is asymptotic to the extent that the covering has to be sufficiently large (in a technical sense) and the functions u^i and u^j have to agree sufficiently closely on $\Omega^i \cap \Omega^j$. In our context Angenent's methods can be applied to glue the single bump solutions u_ε^b together for sufficiently small ε, using a covering that grows as $\varepsilon \to 0$, but the covering cannot grow sufficiently fast to apply Angenent's method if we hold the set

B fixed. We have to drop increasingly large subsets of B as $\varepsilon \to 0$ so that we lose the phenomenon of concentration.

Throughout the rest of this paper, we denote $L^p(\mathbb{R}^n)$ by L^p, the Sobolev space $H^k(\mathbb{R}^n)$ by H^k, the Banach space of k times differentiable functions (with bounded derivatives) $C^k(\mathbb{R}^n)$ by C^k and the Hölder space $C^{k,\lambda}(\mathbb{R}^n)$ by $C^{k,\lambda}$. We usually write C^λ for $C^{0,\lambda}$. When we say that a function is C^k or C^∞, or refer to a C^k or C^∞ function, using the symbols as adjectives, we intend no assumption that the derivatives are bounded.

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2 Assumptions of the problem

We present a generous list of assumptions which are not entirely independent of each other. The justification for them will be developed in the sequel to this paper.

Assumptions on F and V. We assume that F is a C^∞ function of $(a, u) \in I \times \mathbb{R}^n$ where $I \subset \mathbb{R}$ is an open interval. We assume that V is C^∞ on $\mathbb{R}^n$ with range in I. We make the assumption:

(V) V has bounded derivatives with respect to x up to order $\max(3, 2 + \frac{n}{2})$.

We introduce the following positivity condition.

(P) There exists $h > 0$ such that $\dfrac{\partial F}{\partial u}(a, 0) > h$ for all $a \in I$.

Properties of the set of critical points B. We let B be a set consisting of non-degenerate critical points of the function V. We make the following assumptions.

(B1) $\overline{V(B)} \subset I$ (note that $\overline{V(B)}$ is compact by assumption (V)).

(B2) The Hesse matrices $H(b) := D^2V(b)$ are uniformly invertible, that is, there exists M such that $\|H(b)^{-1}\| < M$ for all $b \in B$. Note that this may also be expressed in two further ways as follows: there exists $c > 0$ such that all eigenvalues λ of $H(b)$ for $b \in B$ satisfy $|\lambda| > c$; or most simply that there exists $\gamma > 0$ such that $|\det H(b)| > \gamma$ for all $b \in B$.

Lemma 1. *Under the above conditions the set B has the property: there exists $d > 0$ such that $|b_1 - b_2| > d$ for all $b_1, b_2 \in B$ with $b_1 \neq b_2$.*

Proof. Let $b \in B$. Since $\nabla V(b) = 0$ we have for all x

$$\nabla V(x) = \left(\int_0^1 H(b + t(x - b))\, dt \right) (x - b)$$

If $\nabla V(x) = 0$ and $x \neq b$ then $\int_0^1 H(b + t(x - b))\,dt$ has the eigenvalue 0. Now by the uniform invertibility of $H(b)$ there exists $r > 0$ such that if $b \in B$ and T is a symmetric matrix with $||T - H(b)|| < r$ then T is invertible. Since V has bounded third derivatives the matrix function H is uniformly continuous. Hence there exists $d > 0$ such that if $b \in B$ and $|x - b| < d$ then

$$\left\| \int_0^1 H(b + t(x - b))\,dt - H(b) \right\| < r.$$

It follows that if $b \in B$ and $0 < |x - b| < d$ then $\nabla V(x)$ cannot be 0. $\square$

Definition. A family of functions $(v_j)_{j \in J}$ is said to have **uniform exponential decay** if there exist positive constants C and μ, independent of j in J such that

$$|v_j(x)| \leq Ce^{-\mu|x|}$$

holds for every x in $\mathbb{R}^n$, and every index j.

Throughout the remainder of this section λ is a fixed number in the range $0 < \lambda < 1$.

Properties of the ground state ϕ_a. For each $a \in I$ we assume that there exists a C^∞ solution ϕ_a in the space H^2 to the equation $-\Delta u + F(a, u) = 0$ with the following properties.

(Φ1) The function ϕ_a is radially symmetric. We write $\phi_a(x) = \Phi_a(r)$.

(Φ2) The integral $\displaystyle\int \frac{\partial F}{\partial a}(a, \Phi_a(r))\Phi_a'(r) r\,dx$ is not zero (see assumption (Φ5) below for the existence of this integral).

(Φ3) ϕ_a is a quasi-non-degenerate solution, that is, the operator

$$-\Delta + \frac{\partial F}{\partial u}(a, \phi_a) : C^{2,\lambda} \to C^\lambda$$

has as its kernel the space spanned by the n partial derivatives of ϕ_a, which by radiality are linearly independent, and its range is the subspace of C^λ which is L^2-orthogonal to the kernel (see assumption (Φ4) for why this operator acts in these spaces).

(Φ4) $\phi_a \in C^5(\mathbb{R}^n)$, the map $a \mapsto \phi_a$ is continuous from I to $C^5(\mathbb{R}^n)$, and from I to H^2. Recalling that $\overline{V(B)}$ is a compact set, this implies that there is a constant C independent of a in $\overline{V(B)}$ such that $\|\phi_a\|_{C^5} \leq C$ for $a \in \overline{V(B)}$. The same holds for the H^2 norm.

(Φ5) For each compact $K \subset I$ and for $|\alpha| \leq 4$ the families $(D^\alpha \phi_a)_{a \in K}$ have uniform exponential decay.

Single-bump solution u^b_ε. For this set of assumptions a partial justification can be found in [8]. For every $b \in B$, we assume the existence of a solution u^b_ε which concentrates at the point b, and exists for $\varepsilon < \varepsilon_0$. We want ε_0 to be independent of b. In more detail we make the following assumptions.

(U1) There exists $\varepsilon_0 > 0$ such that for every $b \in B$ and $0 < \varepsilon < \varepsilon_0$, there is a C^∞ solution in H^2 to $-\Delta u + F(V(\varepsilon x), u) = 0$ of the form

$$u^b_\varepsilon(x) = \phi_{V(b)}\left(x - \frac{b}{\varepsilon} + s^b_\varepsilon\right) + \varepsilon^2 w^b_\varepsilon\left(x - \frac{b}{\varepsilon} + s^b_\varepsilon\right) \tag{3}$$

where $s^b_\varepsilon \in \mathbb{R}^n$ and $w^b_\varepsilon \in C^{2,\lambda}$ is orthogonal in L^2 to the partial derivatives of $\phi_{V(b)}$.

(U2) The family of maps $\varepsilon \mapsto s^b_\varepsilon$, $b \in B$, is equicontinuous on the interval $0 < \varepsilon < \varepsilon_0$. As ε tends to 0, s^b_ε tends to 0 uniformly with respect to $b \in B$, and w^b_ε converges in C^λ_{loc} to a function η^b, also uniformly with respect to b. This function η^b belongs to $C^{2,\lambda}$ and is the unique solution in $C^{2,\lambda}$ to the linear problem

$$-\Delta v + \frac{\partial F}{\partial u}(V(b), \phi_{V(b)}(x))v = -\frac{1}{2}\frac{\partial F}{\partial a}(V(b), \phi_{V(b)}(x))\, H(b)x \cdot x \tag{4}$$

that is orthogonal in L^2 to the partial derivatives of $\phi_{V(b)}$.

(U3) The solution u^b_ε belongs to $C^{4,\lambda}$ and there exist bounds on $\|u^b_\varepsilon\|_{C^{4,\lambda}}$ and $\|w^b_\varepsilon\|_{C^1}$ which depend neither on $\varepsilon < \varepsilon_0$ nor on b in B. There is also a bound on $\|\eta^b\|_{C^\lambda}$ which does not depend on b.

(U4) The families $D^\alpha u^b_\varepsilon(x + \xi^b_\varepsilon)$ for $|\alpha| \le 3$ have uniform exponential decay with respect to $b \in B$ and $\varepsilon < \varepsilon_0$. The vector ξ^b_ε, for $b \in B$ and small ε, is a useful abbreviation and is defined by $\xi^b_\varepsilon = \frac{b}{\varepsilon} - s^b_\varepsilon$.

The existence of a solution u^b_ε in H^2, for each b, with the given asymptotic structure (with the limit $w^b_\varepsilon \to \eta^b$ in H^2) was proved in [8]. The assumptions here of regularity, uniformity and decay are additions which will be justified in a subsequent paper.

3 Three elementary lemmas

The following lemmas are elementary. They will be used repeatedly in the proof of the main theorem. Sequences of functions will often be indexed by ν running over the positive integers ("n" is reserved for the dimension of $\mathbb{R}^n$).

The first lemma concerns operators formed by composition (Nemytskii-type operators).

Lemma 2. *Let $f : \mathbb{R}^r \to \mathbb{R}$ be C^∞ and let $u_\nu : \mathbb{R}^n \to \mathbb{R}^r$ be a bounded family in the Banach space $C^{k,\lambda}$ (of r-vector valued functions). Then the family $f \circ u_\nu$ is bounded in*

$C^{k,\lambda}$. *If* $u_\nu \to u$ *in* $C^{k,\lambda}$ *then* $f \circ u_\nu \to f \circ u$ *in* $C^{k,\lambda}$. *If* $u_\nu \to u$ *in* $C^{k,\lambda}_{\mathrm{loc}}$ *then* $f \circ u_\nu \to f \circ u$ *in* $C^{k,\lambda}_{\mathrm{loc}}$.

The second lemma concerns converting local convergence into global convergence.

Lemma 3. *Let* u_ν *be a bounded family in* C^λ *and let* $u_\nu \to u$ *in* C^λ_{loc}. *Let* v_ν *be a family of* C^1 *functions such that* $|v_\nu(x)| + |\nabla v_\nu(x)|$ *tends to* 0 *at infinity uniformly with respect to* ν. *Then* $(u_\nu - u)v_\nu \to 0$ *in* C^λ.

The third lemma concerns inferring a limit equation from a limit relation. The function F was defined in section 2.

Lemma 4. *Let* m_ν, u_ν *and* v_ν *be bounded sequences in* C^λ. *Let us assume that they tend respectively in* C^λ_{loc} *to* m, u *and* v. *If*

$$-\Delta v_\nu + \frac{\partial F}{\partial u}(m_\nu, u_\nu)v_\nu \longrightarrow 0 \quad \text{in } C^\lambda$$

then v *satisfies the limit equation*

$$-\Delta v + \frac{\partial F}{\partial u}(m, u)v = 0$$

in the sense of distributions.

4 Series over the set B

In this section, we see that the uniform exponential decay (U4) of u_ε allows us to construct infinite series such as $\sum_{b\in B} u_\varepsilon^b$ which converge for every sufficiently small ε. The property of the set B given in lemma 1, showing that there exists a positive lower bound d for the distance between distinct points of B, is crucial. As a preliminary, we first prove a lemma concerning numerical series.

Lemma 5. *Let* $0 < \varepsilon < 1$, $\mu > 0$, $b_0 \in B$ *(which may depend on* ε*). Then we have*

$$\sum_{b\in B} e^{-\mu|b-b_0|/\varepsilon} = 1 + \sum_{b\neq b_0} e^{-\mu|b-b_0|/\varepsilon} \leq 1 + Ce^{-\mu d/2\varepsilon}$$

where C *depends only on* n, μ *and* d. *The number* $d > 0$ *is defined in lemma 1.*

Proof. We rewrite the sum over $b \neq b_0$ as

$$\begin{aligned} \sum_{b\neq b_0} e^{-\mu|b-b_0|/\varepsilon} &= \sum_{k\geq 1} \sum_{dk\leq|b-b_0|<d(k+1)} e^{-\mu|b-b_0|/\varepsilon} \\ &\leq \sum_{k\geq 1} e^{-\mu dk/\varepsilon} \operatorname{card}\{b \in B \,:\, |b-b_0| \leq d(k+1)\}. \end{aligned}$$

Since d is a lower bound for the distance between distinct elements of B, any ball of radius $d/2$ contains at most one element of B. Therefore the cardinality occurring above is smaller than the number of balls of radius $d/2$ needed to cover a ball of radius $d(k+1)$. According to results in [17], we obtain

$$\sum_{b\neq b_0} e^{-\mu|b-b_0|/\varepsilon} \leq C \sum_{k\geq 1} k^n e^{-\mu dk/\varepsilon}$$

where C depends only on n. It follows from elementary calculations that

$$k^n e^{-\mu dk/\varepsilon} \leq \left(\frac{2n}{\mu d}\right)^n e^{-n} e^{-\mu dk/2\varepsilon}$$

for every k and n, so that

$$\sum_{b\neq b_0} e^{-\mu|b-b_0|/\varepsilon} \leq C' \sum_{k\geq 1} e^{-\mu dk/2\varepsilon}$$

where C' depends only on n, μ and d. Summing the geometric series gives the lemma. $\square$

The following elementary lemma allows us to study function series as well.

Lemma 6. *Let x in $\mathbb{R}^n$, $0 < \varepsilon < 1$. There exists $b_0 = b_0(x,\varepsilon)$ in B such that*

$$\left|x - \frac{b}{\varepsilon}\right| \geq \frac{1}{2}\frac{|b-b_0|}{\varepsilon}$$

holds for every b in B. Moreover, if x is constrained to a compact subset K of $\mathbb{R}^n$, then $b_0(x,\varepsilon)$ takes only finitely many distinct values.

Proof. We choose b_0 so that $\inf_{b\in B}|x-b/\varepsilon|$ is attained at $b=b_0$. Such an element exists because the set B is discrete. The inequality follows now from the triangle inequality. $\square$

The main convergence result is the following. Recall that ε_0 was defined in assumption (U1) and $\xi^b_\varepsilon = b/\varepsilon - s^b_\varepsilon$ in (U4).

Lemma 7. *Let $\mu > 0$ and $0 < \varepsilon < \varepsilon_1$ where $\varepsilon_1 < \varepsilon_0$. Then the sum $\sum_{b\in B} e^{-\mu|x-\xi^b_\varepsilon|}$ converges in L^∞_{loc} to a bounded continuous function. Moreover, there is a bound on its L^∞-norm which does not depend on ε.*

Proof. Let us first note that by assumption (U2) s^b_ε is bounded for $\varepsilon < \varepsilon_1$ and $b \in B$. For $0 < \varepsilon < \varepsilon_1$ we have that the sum $\sum_{b\in B} e^{-\mu|x-\xi^b_\varepsilon|}$ is bounded by a constant times the sum $\sum_{b\in B} e^{-\mu|x-b/\varepsilon|}$. We may then drop s^b_ε and consider the sum $\sum_{b\in B} e^{-\mu|x-b/\varepsilon|}$. To prove the convergence of this series, let us fix $\varepsilon < \varepsilon_1$ and K a compact subset of $\mathbb{R}^n$. For $x \in K$, let $b_0 = b_0(x,\varepsilon)$ be an element in B given by Lemma 6. As mentioned above, as long as

x remains in K, there are only a finite number of such elements b_0. Choose N_0 so that they are all contained in the open ball of radius N_0 about 0. For $N > N_0$, we have

$$\sum_{b\in B,|b|\geq N} e^{-\mu|x-b/\varepsilon|} \leq \sum_{b\in B,|b|\geq N} e^{-\mu|b-b_0|/2\varepsilon}$$

according to Lemma 6. By choice of N_0, $|b| \geq N$ implies that $b \neq b_0(x,\varepsilon)$ for every $x \in K$ and ε. Therefore Lemma 5 implies

$$\sum_{b\in B,|b|\geq N} e^{-\mu|x-b/\varepsilon|} \leq C' e^{-\mu d/4\varepsilon}$$

where C' depends only on n, μ and d. This shows that the series $\sum_{b\in B} e^{-\mu|x-b/\varepsilon|}$ converges locally uniformly in $\mathbb{R}^n$. The series defines a continuous function, and

$$\sum_{b\in B} e^{-\mu|x-b/\varepsilon|} \leq 1 + C' e^{-\mu d/4\varepsilon}$$

by the same lemma. We therefore obtain a bound on the L^∞-norm of the series $\sum_{b\in B} e^{-\mu|x-b/\varepsilon|}$ which is independent of ε. □

From this we obtain the convergence of some function series to be studied in the next section.

Theorem 8. *Let $\varepsilon_1 < \varepsilon_0$. Under assumptions (U1–4) and (Φ1–5), given a non-empty subset B_0 of B, and $\varepsilon < \varepsilon_1$ the two series $\sum_{b\in B_0} u^b_\varepsilon(x)$ and $\sum_{b\in B_0} \nabla\phi_{V(b)}\big(x-\xi^b_\varepsilon\big)$ converge in C^3_{loc} to functions in $C^3(\mathbb{R}^n)$. Moreover, there is a bound on their C^3 norm which is independent of ε and of B_0.*

Lemma 9. *Let $\mu > 0$ and let K be a compact subset of $\mathbb{R}^n$. Then there exists C such that the inequality*

$$\sum_{b\neq b_0} e^{-\mu|x-(b-b_0)/\varepsilon|} \leq C e^{-\mu d/2\varepsilon}$$

holds for all $x \in K$, $b_0 \in B$ and $\varepsilon \in]0,1[$.

Proof. We use the triangle inequality to obtain

$$\sum_{b\neq b_0} e^{-\mu|x-(b-b_0)/\varepsilon|} \leq e^{\mu|x|} \sum_{b\neq b_0} e^{-\mu|b-b_0|/\varepsilon}.$$

Since $e^{\mu|x|}$ is bounded on K, the result follows from Lemma 5. □

These results allow us to understand how the series $\sum u_\varepsilon^b$ behaves under translations as $\varepsilon \to 0$. It turns out there are two cases that are crucial for the proof of the main theorem. In the first the translation vectors are assumed to diverge from the family of vectors ξ_ε^b as $\varepsilon \to 0$. In the second they are assumed to remain close to it. These two types of translations are subsumed under cases 1 and 2 of the ensuing lemma. We also replace the set B by a subset of B that varies with ε. This is a key idea that will enable us to obtain results uniform with respect to arbitrary subsets of B.

Lemma 10. *Let $\varepsilon_\nu > 0$ be a sequence tending to 0, and for each ν let B_ν be a non-empty subset of B.*

1. *Let x_ν be a sequence in $\mathbb{R}^n$ such that $\inf_{b \in B_\nu} |x_\nu - \xi_{\varepsilon_\nu}^b|$ tends to infinity as $\nu \to \infty$. Then*
$$\sum_{b \in B_\nu} u_{\varepsilon_\nu}^b(x + x_\nu) \to 0 \text{ in } C_{\mathrm{loc}}^\lambda$$
as ν goes to infinity.
2. *Let b_ν be an element of B_ν for every ν. Then there exists a in $\overline{V(B)}$ such that, after restricting to a subsequence,*
$$\sum_{b \in B_\nu} u_{\varepsilon_\nu}^b\left(x + \xi_{\varepsilon_\nu}^{b_\nu}\right) \to \phi_a \text{ in } C_{\mathrm{loc}}^\lambda$$
as $\nu \to \infty$. The element a is an accumulation point of the sequence $V(b_\nu)$.

Proof. Case 1. Let $b_0 = b_0(\nu)$ such that the infimum of $|x_\nu - \xi_{\varepsilon_\nu}^b|$ over $b \in B_\nu$ is attained at $b = b_0$. Then
$$|x_\nu - \xi_{\varepsilon_\nu}^b| \geq \frac{1}{2}|\xi_{\varepsilon_\nu}^b - \xi_{\varepsilon_\nu}^{b_0}|$$
holds for every b in B_ν. Let K be a compact subset of $\mathbb{R}^n$, and x in K. We have
$$\begin{aligned}
\sum_{b \in B_\nu} u_{\varepsilon_\nu}^b(x + x_\nu) &= u_{\varepsilon_\nu}^{b_0}(x + x_\nu) + \sum_{b \neq b_0} u_{\varepsilon_\nu}^b(x + x_\nu) \\
&\leq C(K) e^{-\mu|x_\nu - \xi_{\varepsilon_\nu}^{b_0}|} + C \sum_{b \neq b_0} e^{\mu|x|} e^{-\mu|x_\nu - \xi_{\varepsilon_\nu}^b|} \\
&\leq C(K) e^{-\mu|x_\nu - \xi_{\varepsilon_\nu}^{b_0}|} + C(K) \sum_{b \neq b_0} e^{-\mu|\xi_{\varepsilon_\nu}^b - \xi_{\varepsilon_\nu}^{b_0}|/2}
\end{aligned}$$
where we used the uniform exponential decay, assumption (U4), and the definition of b_0. The constant $C(K)$ depends on the compact set K. By assumption on x_ν, the first term tends to 0 as ν goes to infinity. The second also tends to 0; we dealt with a very similar series in the proof of the previous lemma. The same proof can be carried out for the first derivative, showing that the limit occurs in C_{loc}^λ (in fact C_{loc}^1).

Case 2. Since $V(B)$ is relatively compact and $\overline{V(B)} \subset I$, let us assume, extracting a subsequence if necessary, that $V(b_\nu)$ converges to $a \in I$. Recall that

$$\sum_{b \in B_\nu} u^b_{\varepsilon_\nu}\big(x + \xi^{b_\nu}_{\varepsilon_\nu}\big) = \phi_{V(b_\nu)}(x) + \varepsilon_\nu^2 w^{b_\nu}_{\varepsilon_\nu}(x) + \sum_{b \neq b_\nu} u^b_{\varepsilon_\nu}\big(x + \xi^{b_\nu}_{\varepsilon_\nu}\big).$$

By assumption (U4) and lemma 9, the sum over $b \neq b_\nu$ converges to 0 in C^λ_{loc}. The sequence $\varepsilon_\nu^2 w^{b_\nu}_{\varepsilon_\nu}$ actually converges to 0 in C^λ, since there is a bound on $\|w^b_\varepsilon\|_{C^\lambda}$ which is independent of ε and b (see assumption (U3)). Finally, the functions ϕ_a form a continuum in $C^4(\mathbb{R}^n)$ by assumption (Φ4), so that in particular $\phi_{V(b_\nu)}$ converges to ϕ_a in C^λ. □

Our last two results deal with the interaction of sums and integrals of products of two functions, with uniform exponential decay, translated in different directions.

Lemma 11. *Let $\mu > 0$, $0 < \varepsilon < 1$, and for each ε let $b(\varepsilon)$ be an element of B. Then*

1. $\displaystyle\sum_{b \in B} \sum_{b' \neq b} e^{-\mu|x - b/\varepsilon|} e^{-\mu|x - b'/\varepsilon|} \leq C e^{-\mu d/4\varepsilon}$
2. $\displaystyle\sum_{b \neq b(\varepsilon)} \int_{\mathbb{R}^n} e^{-\mu|x - b/\varepsilon|} e^{-\mu|x - b(\varepsilon)/\varepsilon|}\, dx \leq C' e^{-\mu d/4\varepsilon}.$

The constants C and C' are independent of x, ε and the choice of $b(\varepsilon)$.

Proof. 1. Let x in $\mathbb{R}^n$. Denote the left-hand side by ψ_ε. Let $b_0 = b_0(\varepsilon)$ be given by Lemma 6. We have :

$$\psi_\varepsilon(x) := \sum_{b' \neq b_0} e^{-\mu|x - b_0/\varepsilon|} e^{-\mu|x - b'/\varepsilon|} + \sum_{b \neq b_0} \sum_{b' \neq b} e^{-\mu|x - b/\varepsilon|} e^{-\mu|x - b'/\varepsilon|}$$

In the first term, we use the fact that $e^{-\mu|x - b_0/\varepsilon|}$ is smaller than 1. In the second term, we note that the sum over $b' \neq b$ is smaller than the sum over all possible b', which is uniformly bounded as ε goes to 0, according to Lemma 7, say by the constant C. Hence

$$\psi_\varepsilon(x) \leq \sum_{b' \neq b_0} e^{-\mu|x - b'/\varepsilon|} + C \sum_{b \neq b_0} e^{-\mu|x - b/\varepsilon|} \leq (C+1) \sum_{b \neq b_0} e^{-\mu|x - b/\varepsilon|}.$$

By choice of b_0 we have

$$\psi_\varepsilon(x) \leq (C+1) \sum_{b \neq b_0} e^{-\mu|b - b_0|/2\varepsilon} \leq C' e^{-\mu d/(4\varepsilon)},$$

where C' is independent of x and ε.
2. First change the variable in the integral to obtain

$$\sum_{b \neq b(\varepsilon)} \int_{\mathbb{R}^n} e^{-\mu|x - \frac{b - b(\varepsilon)}{\varepsilon}|} e^{-\mu|x|}\, dx$$

We now fix ε and $b \neq b(\varepsilon)$. For this b the integral is

$$\int_{|x|\leq|b-b(\varepsilon)|/2\varepsilon} e^{-\mu|x-\frac{b-b(\varepsilon)}{\varepsilon}|} e^{-\mu|x|}\,dx + \int_{|x|\geq|b-b(\varepsilon)|/2\varepsilon} e^{-\mu|x-\frac{b-b(\varepsilon)}{\varepsilon}|} e^{-\mu|x|}\,dx$$

In the first integral, we note that $|x - \frac{b-b(\varepsilon)}{\varepsilon}| \geq \frac{|b-b(\varepsilon)|}{2\varepsilon}$. Therefore together the two integrals are smaller than

$$e^{-\mu|b-b(\varepsilon)|/2\varepsilon} \int_{\mathbb{R}^n} e^{-\mu|x|}\,dx \qquad + e^{-\mu|b-b(\varepsilon)|/2\varepsilon} \int_{\mathbb{R}^n} e^{-\mu|x-\frac{b-b(\varepsilon)}{\varepsilon}|}\,dx$$

Using a change of variable, we see that both integrals are equal to $I = \int_{\mathbb{R}^n} e^{-\mu|x|}\,dx$. Hence we obtain

$$2Ie^{-\mu|b-b(\varepsilon)|/2\varepsilon}$$

Going back to the sum over $b \neq b(\varepsilon)$, we see that the whole sum is smaller than

$$2I \sum_{b\neq b(\varepsilon)} e^{-\mu|b-b(\varepsilon)|/2\varepsilon}$$

which is in turn smaller than

$$2ICe^{-\mu d/(4\varepsilon)}$$

with the constant C given by Lemma 5. This ends the proof. $\square$

5 Non-degeneracy at the infinite sum

Throughout this section λ is the Hölder exponent introduced in section 2 and $0 < \varepsilon < \varepsilon_0$ where ε_0 was introduced in assumption (U1).

Let us introduce the non-linear mapping $g_\varepsilon : C^{2,\lambda} \to C^\lambda$ given by

$$g_\varepsilon(u) = -\Delta u + F\big(V(\varepsilon x), u\big).$$

An infinite-bump solution will be obtained by applying the Newton-Kantorovich method to obtain a zero of g_ε near $\sum_{b\in B_0} u^b_\varepsilon$, where B_0 is a non-empty subset of B. The convergence of the sum $\sum_{b\in B_0} u^b_\varepsilon$ is guaranteed by theorem 8. The argument requires us to study the Fréchet derivative of g_ε at the function $\sum_{b\in B_0} u^b_\varepsilon$, that is, the linear operator

$$T^{B_0}_\varepsilon = -\Delta + \frac{\partial F}{\partial u}\Big(V(\varepsilon x), \sum_{b\in B_0} u^b_\varepsilon\Big) : C^{2,\lambda} \to C^\lambda$$

for $\varepsilon < \varepsilon_0$ and $B_0 \subset B$. The next theorem shows that this operator is invertible and controls the norm of the inverse. This is one of the two main results of this paper and its proof will occupy the whole of this section.

Theorem 12. *Make all the assumptions on F, V and B that were detailed in section 2. Let B_0 be an arbitrary non-empty subset of B. For sufficiently small ε, say $\varepsilon < \varepsilon_1$, the operator*

$$T_\varepsilon^{B_0} = -\Delta + \frac{\partial F}{\partial u}\Big(V(\varepsilon x), \sum_{b\in B_0} u_\varepsilon^b\Big)$$

between $C^{2,\lambda}$ and C^λ is invertible, and

$$\|\big(T_\varepsilon^{B_0}\big)^{-1}\|_{\mathcal{L}(C^\lambda, C^{2,\lambda})} \leq C\varepsilon^{-2}$$

with constants $C > 0$ independent of ε and B_0, and ε_1 independent of B_0.

The proof is divided into two distinct arguments, which essentially handle injectivity and surjectivity separately.

Proposition 13. *We assume the same conditions as in Theorem 12. Let ε_ν be a sequence converging to 0, and B_ν be a sequence of non-empty subsets of B, both indexed by the integers. Let v_ν be a bounded sequence in $C^{2,\lambda}$ such that*

$$\varepsilon_\nu^{-2}\, T_{\varepsilon_\nu}^{B_\nu}(v_\nu) = \varepsilon_\nu^{-2}\left(-\Delta + \frac{\partial F}{\partial u}\big(V(\varepsilon_\nu\, x), \sum_{b\in B_\nu} u_{\varepsilon_\nu}^b\big)\right) v_\nu \longrightarrow 0 \quad \text{in } C^\lambda.$$

Then a subsequence of v_ν converges to 0 in $C^{2,\lambda}$.

The proposition implies that for all $\varepsilon < \varepsilon_1$ and $B_0 \subset B$ the operator $T_\varepsilon^{B_0}$ is an isomorphism onto its range and the latter is closed. Moreover, the estimate of Theorem 12 on the inverse holds if we consider the latter as an operator defined on the range. Given Proposition 13, it only remains to prove the surjectivity.

Proposition 14. *We fix $B_0 \subset B$, and ε so that the single-bump solutions all exist. Then the operator $T_\varepsilon^{B_0}$ is surjective.*

Remark. If B_0 is finite, then $T_\varepsilon^{B_0}$ is rather obviously a Fredholm operator of index 0 for every ε, as it is a compact perturbation of $-\Delta + \partial F/\partial u(V(\varepsilon x), 0)$ which is invertible by assumption (P). We then have no need for Proposition 14, since if such an operator is injective, it is also surjective.

5.1 Proof of Proposition 13

Throughout the whole proof, we fix a sequence ε_ν tending to 0, and a sequence of non-empty subsets B_ν of B. We let v_ν be a bounded sequence in $C^{2,\lambda}$ such that

$$\varepsilon_\nu^{-2}\, T_{\varepsilon_\nu}^{B_\nu}(v_\nu) = \varepsilon_\nu^{-2}\left(-\Delta + \frac{\partial F}{\partial u}\big(V(\varepsilon_\nu\, x), \sum_{b\in B_\nu} u_{\varepsilon_\nu}^b\big)\right) v_\nu \longrightarrow 0 \;\text{ in } C^\lambda. \tag{5}$$

We ultimately want to show that a subsequence of v_ν converges to 0 in $C^{2,\lambda}$. Since the sets B_ν are fixed from now on, indices b are implicitly considered to run over B_ν in series such as $\sum u^b_{\varepsilon_\nu}$ and $\sum_{b\neq b_\nu} u^b_{\varepsilon_\nu}$.

We first decompose v_ν in the following way :

$$v_\nu(x) = \sum_{b\in B_\nu} \sigma^b_\nu \cdot \nabla\phi_{V(b)}\big(x - \xi^b_{\varepsilon_\nu}\big) + \alpha_\nu h_\nu, \tag{6}$$

where $\sigma^b_\nu \in \mathbb{R}^n$, $\alpha_\nu \geq 0$, and $h_\nu \in C^{2,\lambda}$ is such that $\|h_\nu\|_{C^{2,\lambda}} = 1$. The choice of h_ν when $\alpha_\nu = 0$ is essentially unimportant, but we could take $h_\nu = 1$. The i-th component of σ^b_ν is defined to be

$$\sigma^b_\nu(i) = \frac{1}{\|\partial\phi_{V(b)}/\partial x_i\|^2_{L^2}} \int_{\mathbb{R}^n} v_\nu(x) \frac{\partial\phi_{V(b)}}{\partial x_i}\big(x - \xi^b_{\varepsilon_\nu}\big)\, dx.$$

The existence of this integral follows from assumption (Φ5) and the fact that v_ν is bounded. Moreover, since $V(b)$ remains in a compact set when b runs over B (and since no ground state ϕ_a is a constant function), assumption (Φ4) implies that there is a positive lower bound on $\|\partial\phi_{V(b)}/\partial x_i\|^2_{L^2}$. Therefore, bearing in mind that v_ν is a bounded sequence in $C^{2,\lambda}$, there is a bound on σ^b_ν which is independent of ν and b. It then follows from theorem 8 that the series in (6) converges in C^3_{loc}. As for the functions $\alpha_\nu h_\nu$, they are asymptotically orthogonal to $\nabla\phi_{V(b_\nu)}(x - \xi^{b_\nu}_{\varepsilon_\nu})$ as the next lemma shows.

Lemma 15. *Let k be a non-negative integer, b_ν an element of B_ν, and $1 \leq i \leq n$. Then, as ν goes to infinity,*

$$\varepsilon_\nu^{-k} \int_{\mathbb{R}^n} \alpha_\nu h_\nu(x) \frac{\partial\phi_{V(b_\nu)}}{\partial x_i}\big(x - \xi^{b_\nu}_{\varepsilon_\nu}\big)\, dx \to 0.$$

Proof. Recall that $\phi_{V(b_\nu)}$ decays at infinity, and that the rate of decay is independent of ν according to assumption (Φ5). Let us fix ν; we use the decomposition (6) and obtain

$$\begin{aligned}\int \alpha_\nu h_\nu(x) \frac{\partial\phi_{V(b_\nu)}}{\partial x_i}\big(x - \xi^{b_\nu}_{\varepsilon_\nu}\big)\, dx &= \int v_\nu(x) \frac{\partial\phi_{V(b_\nu)}}{\partial x_i}\big(x - \xi^{b_\nu}_{\varepsilon_\nu}\big)\, dx \\ &\quad - \sum_{j=1}^n \sigma^{b_\nu}_\nu(j) \int \frac{\partial\phi_{V(b_\nu)}}{\partial x_j}\big(x - \xi^{b_\nu}_{\varepsilon_\nu}\big) \frac{\partial\phi_{V(b_\nu)}}{\partial x_i}\big(x - \xi^{b_\nu}_{\varepsilon_\nu}\big)\, dx \\ &\quad - \sum_{b\neq b_\nu} \int \sigma^b_\nu \cdot \nabla\phi_{V(b)}\big(x - \xi^b_{\varepsilon_\nu}\big) \frac{\partial\phi_{V(b_\nu)}}{\partial x_i}\big(x - \xi^{b_\nu}_{\varepsilon_\nu}\big)\, dx.\end{aligned}$$

The first term on the right-hand side is exactly cancelled out by the term $j = i$ in the sum, by the definition of σ^b_ν. Moreover, in the same sum, the integrals corresponding to $j \neq i$ are all equal to zero, since distinct partial derivatives of the same function $\phi_{V(b_\nu)}$ are orthogonal. Finally, we apply Lemma 11 to obtain that the sum over $b \neq b_\nu$ tends to 0 faster than any power of ε_ν. $\square$

We now plug the decomposition (6) into the relation (5) satisfied by v_ν. This gives

$$\varepsilon_\nu^{-2} \sum_{b \in B_\nu} \left[-\sigma_\nu^b \cdot \Delta\Big(\nabla \phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big)\Big) + \frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu x), \sum_{b' \in B_\nu} u_{\varepsilon_\nu}^{b'}\Big) \sigma_\nu^b \cdot \nabla \phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big) \right] + \alpha_\nu \varepsilon_\nu^{-2} T_{\varepsilon_\nu}^{B_\nu}(h_\nu) \longrightarrow 0 \text{ in } C^\lambda.$$

The summand can be simplified; since ϕ_a satisfies the equation $\Delta \phi_a = F(a, \phi_a)$ for every a, we must have

$$\Delta\Big(\nabla \phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big)\Big) = \frac{\partial F}{\partial u}\Big(V(b), \phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big)\Big) \nabla \phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big)$$

for every ν, b and x. If we take the inner product with σ_ν^b, and set that into the previous limit relation, we obtain

$$\varepsilon_\nu^{-2} \sum_{b \in B_\nu} \left[\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu x), \sum_{b' \in B_\nu} u_{\varepsilon_\nu}^{b'}\Big) - \frac{\partial F}{\partial u}\Big(V(b), \phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big)\Big) \right] \sigma_\nu^b \cdot \nabla \phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big) + \alpha_\nu \varepsilon_\nu^{-2} T_{\varepsilon_\nu}^{B_\nu}(h_\nu) \longrightarrow 0 \quad \text{in } C^\lambda. \tag{7}$$

This serves as the basis for our analysis. We will show that $\varepsilon_\nu^{-2}\alpha_\nu$ is a bounded sequence, and that a subsequence of σ_ν^b tends to 0 uniformly with respect to b in B_ν. In view of the decomposition (6), this is enough to obtain the convergence of v_ν to 0 along a subsequence.

The first step is to show that the sum over B_ν remains bounded in C^λ as ν goes to infinity.

Lemma 16. *As ν goes to infinity,*

$$\varepsilon_\nu^{-2} \sum_{b \in B_\nu} \left[\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu x), \sum_{b' \in B_\nu} u_{\varepsilon_\nu}^{b'}\Big) - \frac{\partial F}{\partial u}\Big(V(b), \phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big)\Big) \right] \sigma_\nu^b \cdot \nabla \phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big) \tag{8}$$

remains bounded in C^λ.

Proof. Let us separate the variations in each argument of F to rewrite (8) as the sum of two terms

$$\begin{aligned} &(I) + (II) \\ &= \varepsilon_\nu^{-2} \sum_b \left[\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu x), \sum_{b' \in B_\nu} u_{\varepsilon_\nu}^{b'}(x)\Big) - \frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu x), \phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big)\Big) \right] \sigma_\nu^b \cdot \nabla \phi_{V(b)}(x - \xi_{\varepsilon_\nu}^b) \\ &+ \varepsilon_\nu^{-2} \sum_b \left[\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu x), \phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big)\Big) - \frac{\partial F}{\partial u}\big(V(b), \phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big)\big) \right] \sigma_\nu^b \cdot \nabla \phi_{V(b)}(x - \xi_{\varepsilon_\nu}^b). \end{aligned}$$

We shall show in fact that both terms remain bounded in C^1, using Taylor's formula to get rid of the factor ε_ν^{-2}. The C^1 norm is much easier to handle than the C^λ norm. Using the asymptotic expansion (3) of $u^b_{\varepsilon_\nu}$, we obtain

$$(I) = \sum_b \int_0^1 \frac{\partial^2 F}{\partial u^2}\Big(V(\varepsilon_\nu x), \phi_{V(b)}\big(x-\xi^b_{\varepsilon_\nu}\big) + t\varepsilon_\nu^2 w^b_{\varepsilon_\nu}\big(x-\xi^b_{\varepsilon_\nu}\big) + t\sum_{b'\neq b} u^{b'}_{\varepsilon_\nu}(x)\Big)\, dt$$
$$\Big(w^b_{\varepsilon_\nu}\big(x-\xi^b_{\varepsilon_\nu}\big) + \varepsilon_\nu^{-2}\sum_{b'\neq b} u^{b'}_{\varepsilon_\nu}(x)\Big)\sigma^b_\nu \cdot \nabla\phi_{V(b)}\big(x-\xi^b_{\varepsilon_\nu}\big).$$

The factor involving the second derivative of F remains bounded in C^1 as ν, b and t vary, by Lemma 2 and the bounds we possess on $w^b_{\varepsilon_\nu}$, (assumption (U3)), and the series $\sum_{b'\neq b} u^{b'}_{\varepsilon_\nu}$ (theorem 8). Recall that $w^b_{\varepsilon_\nu}$ remains bounded in C^1 by assumption (U3), so the term (I) has the form

$$\sum_b f^b_\nu(x)\nabla\phi_{V(b)}\big(x-\xi^b_{\varepsilon_\nu}\big) + \varepsilon_\nu^{-2}\sum_b\sum_{b'\neq b} \widetilde{f}^b_\nu(x) u^{b'}_{\varepsilon_\nu}(x)\,\nabla\phi_{V(b)}\big(x-\xi^b_{\varepsilon_\nu}\big)$$

where f^b_ν and $\widetilde{f}^b_\nu$ are uniformly bounded in C^1. By theorem 8 and lemma 11 we may differentiate these series termwise and then we easily find by the same that the first series is a bounded family in C^1 and the second actually tends to 0 in C^1.

The second term (II) can be rewritten as

$$\sum_b \int_0^1\int_0^1 \frac{\partial^2 F}{\partial a\partial u}\Big(V\big(b+t_1(\varepsilon_\nu x-b)\big), \phi_{V(b)}\big(x-\xi^b_{\varepsilon_\nu}\big)\Big)$$
$$H\big(b+t_1t_2(\varepsilon_\nu x-b)\big)\Big(x-\frac{b}{\varepsilon_\nu}\Big)\cdot\Big(x-\frac{b}{\varepsilon_\nu}\Big)\sigma^b_\nu\cdot\nabla\phi_{V(b)}\big(x-\xi^b_{\varepsilon_\nu}\big)\,t_1\,dt_2\,dt_1$$

using Taylor's formula and the fact that $\nabla V(b) = 0$ for all b. Once again, the factor involving the second derivative of F is bounded in C^1 regardless of ν, b and t. The Hessian matrix H is bounded in C^1 by assumption (V). Finally, since $\xi^b_{\varepsilon_\nu} = b/\varepsilon_\nu - s^b_{\varepsilon_\nu}$, and $s^b_{\varepsilon_\nu}$ tends to 0 uniformly with respect to b, (II) can be expressed as

$$\sum_b\sum_{ijk} g^b_{\nu,ijk}(x)\big(x-\xi^b_{\varepsilon_\nu}\big)_i\big(x-\xi^b_{\varepsilon_\nu}\big)_j\frac{\partial\phi_{V(b)}}{\partial x_k}\big(x-\xi^b_{\varepsilon_\nu}\big)$$

where $g^b_{\nu,ijk}$ is uniformly bounded in C^1 as ν and b vary. The function $\nabla\phi_{V(b)}$ has uniform exponential decay by assumption (Φ5), and so the same is true for $x_ix_j\nabla\phi_{V(b)}$. Applying theorem 8, we see that the term (II) is bounded in C^1, and this concludes the proof. $\square$

We now turn back to the limit relation (7) and proceed to study the sequence α_ν.

Lemma 17. *The sequence $\varepsilon_\nu^{-2}\alpha_\nu$ is bounded.*

Proof. Seeking a contradiction, let us assume that $\varepsilon_\nu^{-2}\alpha_\nu$ is unbounded. In the course of the proof, we shall restrict to a subsequence a finite number of times, usually without change of notation. We begin by going to a subsequence along which $\varepsilon_\nu^2\alpha_\nu^{-1}$ exists and tends to 0. Let us multiply (7) by $\varepsilon_\nu^2\alpha_\nu^{-1}$. The series being bounded in C^λ, according to Lemma 16, we obtain

$$T_{\varepsilon_\nu}^{B_\nu}(h_\nu) = -\Delta h_\nu + \frac{\partial F}{\partial u}\left(V(\varepsilon_\nu x), \sum_b u_{\varepsilon_\nu}^b(x)\right) h_\nu \to 0 \text{ in } C^\lambda. \tag{9}$$

Recall that h_ν was chosen so that $\|h_\nu\|_{C^{2,\lambda}} = 1$ for all ν. We will exhibit a contradiction by showing that h_ν tends to 0 in $C^{2,\lambda}$. We first show that it tends to 0 uniformly, and complete the argument using elliptic regularity. For the first step it is enough to show that, given a subsequence h_{k_ν} of h_ν, and a sequence of vectors x_ν in $\mathbb{R}^n$, a subsequence of $h_{k_\nu}(x_\nu)$ converges to 0. There are two cases.

The case where $\inf_{b\in B_{k_\nu}} \left|x_\nu - \xi^b_{\varepsilon_{k_\nu}}\right|$ remains bounded. Let $b_0 = b_0(\nu)$ be an element of B_{k_ν} where this infimum is attained. Let us translate (9), along the subsequence k_ν to obtain

$$-\Delta h_{k_\nu}\left(x+\xi^{b_0}_{\varepsilon_{k_\nu}}\right)+\frac{\partial F}{\partial u}\left(V\left(\varepsilon_{k_\nu}\left(x+\xi^{b_0}_{\varepsilon_{k_\nu}}\right)\right), \sum_{b\in B_{k_\nu}} u^b_{\varepsilon_{k_\nu}}\left(x+\xi^{b_0}_{\varepsilon_{k_\nu}}\right)\right) h_{k_\nu}\left(x+\xi^{b_0}_{\varepsilon_{k_\nu}}\right) \to 0 \text{ in } C^\lambda.$$

According to Lemma 10, there exists an accumulation point a of the sequence $V(b_0(\nu))$ such that a subsequence of

$$\sum_{b\in B_{k_\nu}} u^b_{\varepsilon_{k_\nu}}\left(x+\xi^{b_0}_{\varepsilon_{k_\nu}}\right)$$

converges to ϕ_a in C^λ_{loc}. Going to this subsequence we may claim that $V\big(\varepsilon_{k_\nu}\big(x+\xi^{b_0}_{\varepsilon_{k_\nu}}\big)\big)$ converges to a in C^λ_{loc}. Finally, since $\|h_\nu\|_{C^{2,\lambda}} = 1$, let us extract a further subsequence along which $h_{k_\nu}(x+\xi^{b_0}_{\varepsilon_{k_\nu}})$ converges to a function g in C^λ_{loc}. Since all the aforementioned sequences are bounded in C^λ, we can apply Lemma 4 to obtain the following equation for g :

$$-\Delta g + \frac{\partial F}{\partial u}(a, \phi_a) g = 0.$$

We now proceed to show that $g = 0$. It is *a priori* in the space C^λ, but an application of the elliptic regularity theorem shows that it is actually in $C^{2,\lambda}$. The assumption that ϕ_a is quasi-non-degenerate therefore implies that g is orthogonal in L^2 to the partial derivatives of ϕ_a. But according to Lemma 15, and by the assumption that $\varepsilon_\nu^{-2}\alpha_\nu$ tends to infinity,

$$\int h_{k_\nu}(x)\,\frac{\partial\phi_{V(b_0(\nu))}}{\partial x_i}\left(x-\xi^{b_0(\nu)}_{\varepsilon_{k_\nu}}\right) dx \to 0$$

for every $1 \le i \le n$, or equivalently

$$\int h_{k_\nu}\left(x + \xi^{b_0(\nu)}_{\varepsilon_{k_\nu}}\right) \frac{\partial \phi_{V(b_0(\nu))}}{\partial x_i}(x)\, dx \to 0.$$

The sequence $h_{k_\nu}\left(x+\xi^{b_0(\nu)}_{\varepsilon_{k_\nu}}\right)$ converges to g in C^λ_{loc}. Moreover, $\partial\phi_{V(b_0(\nu))}/\partial x_i$ converges to $\partial\phi_a/\partial x_i$, and is bounded by a fixed function of the type $Ce^{-\mu|x|}$. Lebesgue's dominated converge theorem implies therefore that

$$\int g(x)\, \frac{\partial \phi_a}{\partial x_i}(x)\, dx = 0$$

for every $1 \le i \le n$. Since we have already shown that g is orthogonal to the partial derivatives of ϕ_a, we obtain $g = 0$. In other words,

$$h_{k_\nu}\left(x + \xi^{b_0}_{\varepsilon_{k_\nu}}\right) \to 0 \text{ in } C^\lambda_{\text{loc}}.$$

By definition of b_0, $|x_\nu - \xi^{b_0}_{\varepsilon_{k_\nu}}|$ remains bounded as ν goes to infinity, so that $h_{k_\nu}(x_\nu)$ tends to 0, which is what we wanted to show.

The case where $\inf_{b\in B_{k_\nu}} |x_\nu - \xi^b_{\varepsilon_{k_\nu}}|$ is unbounded as ν goes to infinity. Our starting point is again (9). Going to a subsequence if necessary, we can assume that $\inf_{b\in B_{k_\nu}} |x_\nu - \xi^b_{\varepsilon_{k_\nu}}|$ actually tends to infinity. Lemma 10 then implies that

$$\sum_{b\in B_{k_\nu}} u^b_{\varepsilon_{k_\nu}}\left(x + x_\nu\right) \to 0 \text{ in } C^\lambda_{\text{loc}}.$$

Moreover, $V(\varepsilon_{k_\nu}(x + x_\nu))$ is a bounded sequence in L^∞, while its derivative converges uniformly to 0. We may hence extract a subsequence along which it converges to a constant c in C^λ_{loc}. Finally, let us extract another subsequence so that $h_{k_\nu}(x + x_\nu)$ converges to a function g in C^λ_{loc}. Lemma 4 then gives the following limit equation :

$$-\Delta g + \frac{\partial F}{\partial u}(c, 0) g = 0.$$

The elliptic regularity theorem tells us that $g \in C^{2,\lambda}$. Since $\partial F/\partial u(c, 0) \ge h > 0$, the operator $-\Delta + \partial F/\partial u(c, 0) : C^{2,\lambda} \to C^\lambda$ is invertible, so that $g = 0$. In other words, $h_{k_\nu}(x + x_\nu)$ tends to 0 in C^λ_{loc} along a subsequence, and in particular $h_{k_\nu}(x_\nu)$ tends to 0.

On the basis of this two-case analysis, we conclude that h_ν tends to 0 uniformly. It remains to show that it does so in $C^{2,\lambda}$ and the contradiction will follow. This is a standard elliptic regularity argument, based on the use of (9). This shows that the sequence $\varepsilon_\nu^{-2}\alpha_\nu$ is bounded. □

Knowing now that the sequence $\varepsilon_\nu^{-2}\alpha_\nu$ is bounded we return to (7) which we recall for convenience

$$\varepsilon_\nu^{-2}\sum_{b\in B_\nu}\left[\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu x),\sum_{b'\in B_\nu}u_{\varepsilon_\nu}^{b'}\Big)-\frac{\partial F}{\partial u}\Big(V(b),\phi_{V(b)}\big(x-\xi_{\varepsilon_\nu}^b\big)\Big)\right]\sigma_\nu^b\cdot\nabla\phi_{V(b)}\big(x-\xi_{\varepsilon_\nu}^b\big)$$
$$+\alpha_\nu\varepsilon_\nu^{-2}T_{\varepsilon_\nu}^{B_\nu}(h_\nu)\longrightarrow 0\ \text{ in } C^\lambda.$$

Our goal now is to show that the sequence σ_ν^b tends to 0 uniformly with respect to b. To achieve this, we introduce an arbitrary sequence b_ν such that $b_\nu\in B_\nu$ for each ν, and show that $\sigma_\nu^{b_\nu}$ tends to 0. Translating the above limit relation gives

$$\begin{aligned}\varepsilon_\nu^{-2}\sum_{b\in B_\nu}\Bigg[&\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x+\xi_{\varepsilon_\nu}^{b_\nu})),\sum_{b'\in B_\nu}u_{\varepsilon_\nu}^{b'}(x+\xi_{\varepsilon_\nu}^{b_\nu})\Big)\\ &-\frac{\partial F}{\partial u}\big(V(b),\phi_{V(b)}(x-\xi_{\varepsilon_\nu}^b+\xi_{\varepsilon_\nu}^{b_\nu})\big)\Bigg]\sigma_\nu^b\cdot\nabla\phi_{V(b)}(x-\xi_{\varepsilon_\nu}^b+\xi_{\varepsilon_\nu}^{b_\nu})\\ +\alpha_\nu\varepsilon_\nu^{-2}&\left(-\Delta+\frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x+\xi_{\varepsilon_\nu}^{b_\nu})),\sum_{b'\in B_\nu}u_{\varepsilon_\nu}^{b'}(x+\xi_{\varepsilon_\nu}^{b_\nu})\Big)\right)h_\nu(x+\xi_{\varepsilon_\nu}^{b_\nu})\longrightarrow 0\text{ in } C^\lambda.\end{aligned}\tag{10}$$

Let σ be an accumulation point of the sequence $\sigma_\nu^{b_\nu}$, and assume, restricting to a subsequence if necessary, that $\sigma_\nu^{b_\nu}$ converges to σ. We may also assume that $\alpha_\nu\varepsilon_\nu^{-2}$ converges to a real number κ thanks to Lemma 17, and that $h_\nu(x+\xi_{\varepsilon_\nu}^{b_\nu})$ converges in C^λ_{loc} to a function g. Finally, we can further assume that $V(b_\nu)$ converges to $a\in I$ while the Hessian matrix $H(b_\nu)$ converges to a matrix A, which is non-degenerate according to (B2).

Lemma 18. *With the assumptions of the last paragraph, as ν tends to infinity, a subsequence of the left-hand side of (10) converges in C^λ_{loc} to*

$$\left[\frac{\partial^2F}{\partial u^2}(a,\phi_a)\rho+\frac{1}{2}\frac{\partial^2F}{\partial a\partial u}(a,\phi_a)Ax\cdot x\right]\sigma\cdot\nabla\phi_a+\kappa\Big(-\Delta+\frac{\partial F}{\partial u}(a,\phi_a)\Big)g,$$

where the function ρ satisfies

$$-\Delta\rho+\frac{\partial F}{\partial u}(a,\phi_a)\rho=-\frac{1}{2}\frac{\partial F}{\partial a}(a,\phi_a)Ax\cdot x.\tag{11}$$

Proof. Let us assume that we have extracted the subsequences described above. We first treat the term containing h_ν in (10). The sequence $\alpha_\nu{\varepsilon_\nu}^{-2}$ converges to κ; $V(b_\nu)$ converges to a, so that $V(\varepsilon_\nu(x+\xi_{\varepsilon_\nu}^{b_\nu}))$ converges in C^λ_{loc} to a and $\sum_{b'\in B_\nu}u_{\varepsilon_\nu}^{b'}(x+\xi_{\varepsilon_\nu}^{b_\nu})$ converges in

C^λ_{loc} to ϕ_a according to Lemma 10. Moreover, $h_\nu(x + \xi^{b_\nu}_{\varepsilon_\nu})$ converges in C^λ_{loc} to g. All the aforementioned sequences are bounded in C^λ. We can therefore apply Lemma 4 to obtain that

$$\alpha_\nu \varepsilon_\nu^{-2}\left(-\Delta + \frac{\partial F}{\partial u}\Big(V(\varepsilon_\nu(x + \xi^{b_\nu}_{\varepsilon_\nu})), \sum_{b' \in B_\nu} u^{b'}_{\varepsilon_\nu}(x + \xi^{b_\nu}_{\varepsilon_\nu})\Big)\right) h_\nu(x+\xi^{b_\nu}_{\varepsilon_\nu}) \to \kappa\Big(-\Delta + \frac{\partial F}{\partial u}(a, \phi_a)\Big) g$$

in C^λ_{loc}.

Let us now deal with the series in (10). It turns out that only the term $b = b_\nu$ in the sum makes a non-zero contribution to the limit. Indeed, the sum over $b \neq b_\nu$ tends to 0 in C^λ_{loc}, as we proceed to show. First of all, each one of the two terms involving the derivatives of F is bounded in C^λ. This follows from lemma 2 and theorem 8 and the properties of V and ϕ_a. The sequence σ^b_ν is likewise bounded uniformly with respect to b. Finally $\nabla\phi_{V(b)}$ has uniform exponential decay at infinity, and hence so does the whole summand. We can therefore apply Lemma 9 to show that the series over $b \neq b_\nu$ converges to 0 in C^λ_{loc}.

Let us now turn to the term $b = b_\nu$ in the series. We write it as the sum of the two terms (I) and (II) as in the proof of Lemma 16, that is :

$$(I) = \int_0^1 \frac{\partial^2 F}{\partial u^2}\Big(V\big(\varepsilon_\nu\big(x + \xi^{b_\nu}_{\varepsilon_\nu}\big)\big), \phi_{V(b_\nu)}(x) + t\varepsilon_\nu^2 w^{b_\nu}_{\varepsilon_\nu}(x) + t\sum_{b' \neq b_\nu} u^{b'}_{\varepsilon_\nu}\big(x + \xi^{b_\nu}_{\varepsilon_\nu}\big)\Big)$$
$$\Big(w^{b_\nu}_{\varepsilon_\nu}(x) + \varepsilon_\nu^{-2}\sum_{b' \neq b_\nu} u^{b'}_{\varepsilon_\nu}\big(x + \xi^{b_\nu}_{\varepsilon_\nu}\big)\Big)\,\sigma^{b_\nu}_\nu \cdot \nabla\phi_{V(b_\nu)}(x)\,dt. \quad (12)$$

The term involving the second derivative of F remains bounded in C^λ (as was seen in the proof of Lemma 16) and converges in C^λ_{loc} to $\partial^2 F/\partial u^2(a, \phi_a(x))$ as ν goes to infinity (for every t in $[0, 1]$). Indeed, $\sum_{b' \neq b_\nu} u^{b'}_{\varepsilon_\nu}\big(x + \xi^{b_\nu}_{\varepsilon_\nu}\big)$ converges to 0 in C^λ_{loc} according to Lemma 10, and $w^{b_\nu}_{\varepsilon_\nu}$ is a bounded sequence in C^λ.

Let us extract a further subsequence along which $w^{b_\nu}_{\varepsilon_\nu}$ converges in C^λ_{loc} to a function ρ. The function ρ will be identified later on. Now $\sigma^{b_\nu}_\nu \cdot \nabla\phi_{V(b_\nu)}$ converges to $\sigma \cdot \nabla\phi_a$ in C^λ, and these functions decay exponentially at infinity, uniformly with respect to ν. It follows by lemma 3 that

$$\Big(w^{b_\nu}_{\varepsilon_\nu} + \varepsilon_\nu^{-2}\sum_{b' \neq b_\nu} u^{b'}_{\varepsilon_\nu}\big(x + \xi^{b_\nu}_{\varepsilon_\nu}\big)\Big)\,\sigma^{b_\nu}_\nu \cdot \nabla\phi_{V(b_\nu)} \to \rho(\sigma \cdot \nabla\phi_a)$$

in C^λ, so that the integrand of (12) converges to

$$\frac{\partial^2 F}{\partial u^2}(a, \phi_a)\rho(\sigma \cdot \nabla\phi_a)$$

in C^λ. Now the C^λ-norm of the integrand is bounded independently of ν and t. We may therefore apply Lebesgue's dominated convergence theorem to obtain that (12) converges to

$$\frac{\partial^2 F}{\partial u^2}(a, \phi_a)\rho(\sigma \cdot \nabla\phi_a)$$

in C^λ.

We apply similar techniques to the term (II). Recall that

$$(II) = \int_0^1 \int_0^1 \frac{\partial^2 F}{\partial a \partial u}\Big(V\big(\varepsilon_\nu\big(x + \xi^{b_\nu}_{\varepsilon_\nu}\big), \phi_{V(b_\nu)}\Big) \\ H\big(b_\nu + t_1 t_2 \varepsilon_\nu (x - s_{\varepsilon_\nu})\big)\,(x - s_{\varepsilon_\nu}) \cdot (x - s_{\varepsilon_\nu})\,\nabla\phi_{V(b_\nu)} \cdot \sigma^{b_\nu}_\nu\, t_1\, dt_1\, dt_2.$$

The term involving the second derivative of F remains bounded in C^λ and tends to $\dfrac{\partial^2 F}{\partial a \partial u}(a, \phi_a)$ in C^λ_{loc}. The matrices $H\big(b_\nu + t_1 t_2 \varepsilon_\nu (x - s_{\varepsilon_\nu})\big)$ converge in C^λ_{loc} to the matrix A, while s_{ε_ν} tends to 0 and $\nabla\phi_{V(b_\nu)} \cdot \sigma^{b_\nu}_\nu$ to $\nabla\phi_a \cdot \sigma$ in C^λ. All the aforementioned sequences are bounded in C^λ as ν, s and t vary. We may therefore apply Lebesgue's dominated convergence theorem and obtain that (II) converges to

$$\frac{1}{2}\frac{\partial^2 F}{\partial a \partial u}(a, \phi_a) A x \cdot x\,(\nabla\phi_a \cdot \sigma)$$

in C^λ_{loc}. This concludes the computation of the limit.

It remains to infer the equation satisfied by ρ. Recall that it is obtained as a limit in C^λ_{loc} of the sequence $w^{b_\nu}_{\varepsilon_\nu}$. Now w^b_ε converges to η^b in C^λ_{loc}, uniformly with respect to b in B. It follows that

$$\rho = \lim_{\nu \to +\infty} w^{b_\nu}_{\varepsilon_\nu} = \lim_{\nu \to +\infty} \eta^{b_\nu},$$

the limits being taken in C^λ_{loc}. The functions η^{b_ν} satisfy the equation

$$\Delta\eta^{b_\nu} + \frac{\partial F}{\partial u}\big(V(b_\nu), \phi_{V(b_\nu)}\big)\eta^{b_\nu} = -\frac{1}{2}\frac{\partial F}{\partial a}\big(V(b_\nu), \phi_{V(b_\nu)}\big) H(b_\nu) x \cdot x$$

which implies, taking the limit as ν goes to infinity,

$$\Delta\rho + \frac{\partial F}{\partial u}(a, \phi_a)\rho = -\frac{1}{2}\frac{\partial F}{\partial a}(a, \phi_a) A x \cdot x.$$

□

We now have shown that, if σ is an accumulation point of the sequence $\sigma^{b_\nu}_\nu$, then there exist $a \in \overline{V(B)}$, $\kappa \in \mathbb{R}$, $g \in C^\lambda$ and an invertible matrix A such that

$$\left[\frac{\partial^2 F}{\partial u^2}(a, \phi_a)\rho + \frac{1}{2}\frac{\partial^2 F}{\partial a \partial u}(a, \phi_a) A x \cdot x\right]\sigma \cdot \nabla\phi_a + \kappa\Big(-\Delta + \frac{\partial F}{\partial u}(a, \phi_a)\Big)g = 0. \tag{13}$$

It follows from an elliptic regularity argument that $\kappa g \in C^{2,\lambda}$. If we differentiate (11) and take the inner-product with σ, we obtain

$$\begin{aligned}
-\Delta(\nabla\rho\cdot\sigma) + \frac{\partial F}{\partial u}(a,\phi_a)\nabla\rho\cdot\sigma + \frac{\partial^2 F}{\partial u^2}(a,\phi_a)\rho\,(\nabla\phi_a\cdot\sigma) = \\
-\frac{\partial F}{\partial a}(a,\phi_a)Ax\cdot x - \frac{1}{2}\frac{\partial^2 F}{\partial a\partial u}(a,\phi_a)Ax\cdot\sigma\,(\nabla\phi_a\cdot\sigma)
\end{aligned}$$

which, combined with (13) gives

$$\left(-\Delta + \frac{\partial F}{\partial u}(a,\phi_a)\right)\big(\kappa g + \nabla\rho\cdot\sigma\big) = -\frac{\partial F}{\partial a}(a,\phi_a)Ax\cdot\sigma.$$

Since $\kappa g + \nabla\rho\cdot\sigma \in C^{2,\lambda}$, we see that the right-hand side belongs to the range of $-\Delta + \partial F/\partial u(a,\phi_a)$ as an operator from $C^{2,\lambda}$ to C^{λ}. By assumption, it must then be orthogonal to the partial derivatives of ϕ_a, and we obtain

$$\int \frac{\partial F}{\partial a}(a,\phi_a(x))Ax\cdot\sigma\,\frac{\partial\phi_a}{\partial x_i}\,dx = 0$$

for $1 \le i \le n$. We now deduce that $\sigma = 0$, using the fact that A is an invertible matrix. If we write $\sigma = (\sigma_1, ..., \sigma_n)$, the inner product $Ax\cdot\sigma$ can be written as

$$Ax\cdot\sigma = \sum_{j,k}\sigma_j x_k A_{j,k}$$

where the $A_{j,k}$ are the coefficients of the matrix A. Since ϕ_a is spherically symmetric, we write $\phi_a(x) = \Phi_a(r)$ with $r = |x|$, and obtain that

$$\sum_{j,k}\sigma_j A_{j,k}\int x_i x_k\,\frac{\partial F}{\partial a}(a,\Phi_a(r))\frac{\Phi_a'(r)}{r}\,dx = 0.$$

Because of spherical symmetry, the integrals involving $x_i x_k$ with $i \neq k$ must vanish. We are then left with

$$\sum_j \sigma_j A_{j,i}\int \frac{\partial F}{\partial a}(a,\Phi_a(r))\Phi_a'(r)\frac{{x_i}^2}{r}\,dx = 0$$

for every $1 \le i \le n$. This integral is independent of i by spherical symmetry, so that

$$\int \frac{\partial F}{\partial a}(a,\Phi_a(r))\Phi_a'(r)\frac{{x_i}^2}{r}\,dx = \frac{1}{n}\int \frac{\partial F}{\partial a}(a,\Phi_a(r))\Phi_a'(r)r\,dx.$$

The integral on the right-hand side is not zero by assumption. It follows therefore that

$$\sum_i \sigma_i A_{j,i} = 0,$$

and since A is invertible, we obtain $\sigma = 0$.

Recall that σ was an arbitrary accumulation point of $\sigma_\nu^{b_\nu}$ which is a bounded sequence. It follows that it must converge to 0. The sequence b_ν was also arbitrary, and σ_ν^b therefore converges to 0 uniformly with respect to b (in B_ν). We also obtained that α_ν converges to 0 in Lemma 17. If we consider again the decomposition of v_ν

$$\sum_{b\in B_\nu} \sigma_\nu^b \cdot \nabla\phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big) + \alpha_\nu h_\nu,$$

we see that $\alpha_\nu h_\nu$ tends to 0 in $C^{2,\lambda}$ since $\|h_\nu\|_{C^{2,\lambda}} = 1$ for every ν. Moreover, since the series $\sum \nabla\phi_{V(b)}\big(x - \xi_{\varepsilon_\nu}^b\big)$ remains bounded in $C^{2,\lambda}$ as ν goes to infinity, and σ_ν^b tends to 0 uniformly with respect to b, it follows that v_ν itself tends to 0 in $C^{2,\lambda}$. This concludes the proof of Proposition 13.

5.2 Proof of Proposition 14

In this subsection, we prove that the operator $T_\varepsilon^{B_0} = -\Delta + \dfrac{\partial F}{\partial u}(V(\varepsilon x), \sum_{b\in B_0} u_\varepsilon^b)$ from $C^{2,\lambda}$ to C^λ is surjective for all subsets B_0 of B, and sufficiently small ε. Its fine structure will not be used, and we simply write $T_\varepsilon^{B_0} = -\Delta + G(x)$. The function G is of class C^∞.

Proof. Let R be the range of $T_\varepsilon^{B_0}$. By proposition 13, it is known to be closed in C^λ. We let S be a linear form on C^λ which vanishes identically on R. It defines a distribution S_0 by restriction; actually S_0 belongs to H^{-k}, where k is the smallest integer larger than $1 + n/2$, according to Morrey's embedding theorem. Since S vanishes on all functions of the form $-\Delta u + G(x)u$, where $u \in C^{2,\lambda}$, it follows that $-\Delta S_0 + G(x)S_0 = 0$ in the sense of distributions.

We now use a bootstrap argument to show that $S_0 = 0$. Because G has k bounded derivatives by assumption (V), (this is where we need V to have bounded derivatives to order $2 + n/2$), $G(x)S_0$ belongs to H^{-k} as well, and $\|G(x)S_0\|_{H^{-k}}$ is smaller than a constant times $\|S_0\|_{H^{-k}}$. It follows that $S_0 \in H^{-k+2}$ by elliptic regularity. We repeat this argument until we obtain that $S_0 \in H^{k+2}$, which is embedded into $C^{2,\lambda}$ by choice of k. The distribution S_0 is therefore a classical solution of $-\Delta u + G(x)u = 0$; since the operator $-\Delta + G(x)$ is injective from $C^{2,\lambda}$ to C^λ by Proposition 13, it follows that $S_0 = 0$.

In other words, every linear form on C^λ which vanishes on R is identically zero as a distribution. It follows by the Hahn-Banach theorem that R contains the C^∞ functions with compact support. We now prove that $R = C^\lambda$ by a standard truncation and regularization argument. Let $f \in C^\lambda$. We let ψ_ν be a sequence of C^∞ functions, supported in the ball about 0 of radius $\nu + 1$, and identically equal to 1 on the ball of radius ν. We also define φ_ν to be an approximation of unity, supported in the ball of radius $1/\nu$. We approximate f by the sequence $f_\nu = \psi_\nu(\varphi_\nu * f)$. Clearly this is a C^∞ function with compact support, and hence it belongs to the range R. For every ν, there exists therefore a unique $v_\nu \in C^\lambda$ such that

$$-\Delta v_\nu + G(x) v_\nu = f_\nu.$$

According to proposition 13, the operator $-\Delta + G(x)$ has a bounded inverse from R to $C^{2,\lambda}$ so that

$$\|v_\nu\|_{C^{2,\lambda}} \leq C \|\psi_\nu\|_{C^\lambda} \, \|\varphi_\nu * f\|_{C^\lambda}.$$

The constant C is independent of ν (we work at fixed ε here). We can choose ψ_ν so that $\|\psi_\nu\|_{C^\lambda}$ is always smaller than 3. Since φ_ν is an approximation of unity, it follows that $||\varphi_\nu * f||_{C^\lambda}$ is bounded by a constant times $||f||_{C^\lambda}$. This shows that v_ν is a bounded sequence in $C^{2,\lambda}$, and, extracting a subsequence if necessary, we may assume that it converges in C^2_{loc} to a function v. The latter belongs to $C^{2,\lambda}$ by elliptic regularity. It is easily seen that v is the inverse image of f under the operator $-\Delta + G(x)$, which is therefore surjective. □

6 Infinite-bump solutions

Non-degeneracy at the infinite series $\sum_{b \in B_0} u_\varepsilon^b$ is used to construct solutions with infinitely many bumps.

Theorem 19. *There exists $\varepsilon_1 > 0$ such that, if $\varepsilon < \varepsilon_1$ and B_0 is a non-empty subset of B, there exists a solution $U_\varepsilon^{B_0} \in C^{2,\lambda}$ to the equation $-\Delta u + F(V(\varepsilon x), u) = 0$ such that*

$$\left\| U_\varepsilon^{B_0} - \sum_{b \in B_0} u_\varepsilon^b \right\|_{C^{2,\lambda}} \leq C e^{-\sigma/\varepsilon}.$$

The constants C and σ are independent of ε and B_0.

Proof. The proof is based on the famous Newton-Kantorovich method. We define an operator $g_\varepsilon^{B_0}$ from $C^{2,\lambda}$ to C^λ by $g_\varepsilon^{B_0}(u) = -\Delta u + F(V(\varepsilon x), u)$. According to theorem 12, $\psi_\varepsilon = \sum_{b \in B_0} u_\varepsilon^b$ is a regular point of $g_\varepsilon^{B_0}$, and the norm of the inverse of the derivative at that point is controlled by $C\varepsilon^{-2}$. Moreover, the ψ_ε form a bounded set in $C^{2,\lambda}$ as ε tends to 0, say $\|\psi_\varepsilon\| \leq R$. We define

- $\eta_\varepsilon = \|Dg_\varepsilon(\psi_\varepsilon)^{-1} \, g_\varepsilon(\psi_\varepsilon)\|_{C^{2,\lambda}}$, and
- $K_\varepsilon = \|Dg_\varepsilon(\psi_\varepsilon)^{-1}\| \sup \left\{ \|D^2 g_\varepsilon(u)\| \, : \, \|u\|_{C^{2,\lambda}} \leq R + 1 \right\}$.

Then, provided

$$h_\varepsilon := \eta_\varepsilon K_\varepsilon < \frac{1}{2}, \tag{14}$$

$$r_\varepsilon := \frac{1 - \sqrt{1 - 2h_\varepsilon}}{h_\varepsilon} \eta_\varepsilon \leq R \tag{15}$$

there exists a unique zero $U_\varepsilon^{B_0}$ of $g_\varepsilon^{B_0}$ such that $\|U_\varepsilon^{B_0} - \sum_{b\in B_0} u_\varepsilon^b\|_{C^{2,\lambda}} \le r_\varepsilon$. Here we have slightly adapted one of many readable accounts of Newton-Kantorovich [13] to our needs.

We proceed to show that the inequalities (14) and (15) are satisfied for every sufficiently small ε and B_0 and that $r_\varepsilon \le Ce^{-\sigma/\varepsilon}$ for some constants C and σ. This implies the theorem.

We first estimate $g_\varepsilon(\psi_\varepsilon)$. Since each individual u_ε^b is a solution of $-\Delta u + F(V(\varepsilon x), u) = 0$, we have :

$$g_\varepsilon(\psi_\varepsilon) = g_\varepsilon\Big(\sum u_\varepsilon^b\Big) = F\Big(V(\varepsilon x), \sum_b u_\varepsilon^b\Big) - \sum_b F\big(V(\varepsilon x), u_\varepsilon^b\big).$$

Let us enumerate the elements in B_0 by $(b_k)_{k\in\mathbb{N}}$. We rewrite the right-hand side as a telescoping series

$$\sum_{k\ge 0}\left[F\Big(V(\varepsilon x), \sum_{l\ge k} u_\varepsilon^{b_l}\Big) - F(V(\varepsilon x), u_\varepsilon^{b_k}) - F\Big(V(\varepsilon x), \sum_{l\ge k+1} u_\varepsilon^{b_l}\Big) + F(V(\varepsilon x), 0)\right].$$

Recall that $F(a,0) = 0$ for every a. Applying Taylor's formula, we find that this is equal to

$$\sum_{k\ge 0}\sum_{m\ge k+1}\int_0^1\int_0^1 \frac{\partial^2 F}{\partial u^2}\Big(V(\varepsilon x), s u_\varepsilon^{b_k} + t\sum_{l\ge k+1} u_\varepsilon^{b_l}\Big)\, u_\varepsilon^{b_k} u_\varepsilon^{b_m}\, ds\, dt.$$

We now use a consequence of the uniform exponential decay of the functions u_ε^b, namely Lemma 11, to obtain that

$$\|g_\varepsilon(\psi_\varepsilon)\|_{C^\lambda} \le Ce^{-\sigma/\varepsilon}$$

where C and σ are independent of ε and B_0. Since the inverse of the derivative of g_ε at ψ_ε blows up like ε^{-2}, we certainly have $h_\varepsilon < 1/2$ for all sufficiently small ε, that is, inequality (14). Inequality (15) immediately follows (increasing the constant C if necessary), as well as the fact that r_ε tends to 0 exponentially fast. □

Since F and V are C^∞, it actually follows that $U_\varepsilon^{B_0}$ is also C^∞.

Theorem 20. *If the functions u_ε^b are all positive, then so is the infinite-bump solution $U_\varepsilon^{B_0}$ for every B_0 and every sufficiently small ε.*

Proof. First of all, since U_ε approaches the positive function $\sum_{b\in B_0} u_\varepsilon^b$ uniformly we must have $U_\varepsilon(x) \ge -Ce^{-\sigma/\varepsilon}$ for all x. Let us rewrite the equation $-\Delta U_\varepsilon + F(V(\varepsilon x), U_\varepsilon) = 0$ as

$$-\Delta U_\varepsilon + G_\varepsilon(x) U_\varepsilon = 0,$$

where

$$G_\varepsilon(x) = \int_0^1 \frac{\partial F}{\partial u}(V(\varepsilon x), tU_\varepsilon)\, dt.$$

We see that $G_\varepsilon(x) > h/2 > 0$ whenever $U_\varepsilon(x)$ is negative and ε is small enough, say smaller than ε_2, which is independent of B_0. The positive constant h is given by the positivity assumption

$$\frac{\partial F}{\partial u}(V(\varepsilon x), 0) > h.$$

We now fix B_0 and $\varepsilon < \varepsilon_2$ throughout the rest of the proof, and we drop the corresponding subscripts.

Let us first show that $U = U_\varepsilon$ is non-negative. The proof is much simpler when 0 is a regular value of U. Now this is not always the case, but according to Sard's theorem (recall that U is C^∞), there exists a sequence δ_ν of regular values of U_ε approaching 0 from below. We define

$$\Omega_\nu = \{x \in \mathbb{R}^n \,,\, U(x) < \delta_\nu\} \quad \text{and } v_\nu := (U - \delta_\nu)^- = \min(0, U - \delta_\nu).$$

Since δ_ν is a regular value, the boundary of Ω_ν is a smooth submanifold of $\mathbb{R}^n$. We now fix an integer ν, and show that the tempered distribution $-\Delta v_\nu + G(x)v_\nu$ is positive.

To do so, we let φ be a positive test function. Green's second identity on Ω_ν gives

$$\int_{\Omega_\nu} (U - \delta_\nu)\Delta\varphi \, dx = \int_{\Omega_\nu} (\Delta U)\varphi \, dx + \int_{\partial\Omega_\nu} \left((U_\varepsilon - \delta_\nu)\frac{\partial \phi}{\partial n} - \varphi \frac{\partial U}{\partial n} \right) d\sigma.$$

Here $d\sigma$ is the surface element on $\partial\Omega_\nu$ and $\partial/\partial n$ denotes the outer normal derivative on the boundary. Since $U = \delta_\nu$ on $\partial\Omega_\nu$, the first boundary term vanishes; moreover, the outer normal derivative of U must be greater than 0, as we go from a region where $U < \delta_\nu$ to one where $U > \delta_\nu$. It follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} -v_\nu \Delta\varphi + G(x)v_\nu\varphi \, dx &= \int_{\Omega_\nu} -(U - \delta_\nu)\Delta\varphi + G(x)(U - \delta_\nu \, \varphi) \, dx \\
&= \int_{\Omega_\nu} \big(- \Delta U + G(x)U \big)\varphi \, dx - \delta_\nu \int_{\Omega_\nu} G(x)\varphi \, dx + \int_{\partial\Omega_\nu} \varphi \frac{\partial U}{\partial n} \, d\sigma.
\end{aligned}$$

This is greater than 0 since $-\Delta U + G(x)U = 0$ holds pointwise. Recall that, by choice of ε_2, G is positive where U is negative, in particular in Ω_ν. It follows that $-\Delta v_\nu + G(x)v_\nu$ is a positive distribution. Since $G(x) > h/2$ wherever v_ν does not vanish, we obtain in particular that the tempered distribution $-\Delta v_\nu + (h/2)v_\nu$ is positive. We denote it by T. It acts as follows on the test function φ :

$$\langle T, \varphi \rangle = \int_{\mathbb{R}^n} \psi(x) \, \varphi(x) \, dx + \int_{\partial\Omega_\nu} \varphi(y) d\mu(y),$$

where ψ is a positive bounded function, and μ a positive measure on $\partial\Omega_\nu$.

Let E be the fundamental solution of the Helmholtz equation $-\Delta u + (h/2)u = 0$. The distribution equation $-\Delta u + (h/2)u = T$ has a unique tempered solution, which is given by the convolution of E with T. This convolution is well-defined, since T is tempered, and E decays exponentially, while being in L^1, see [16] and [15]. Therefore v_ν must be equal to $E * T$, so that

$$v_\nu(x) = \int_{\mathbb{R}^n} E(x-y)\psi(y)\,dy + \int_{\partial\Omega_\nu} E(x-y)\,d\mu(y).$$

Since E and ψ are positive functions, and μ is a positive measure, it follows that $v_\nu \geq 0$ for every ν. Since v_ν is negative by definition, it follows that $v_\nu = 0$ everywhere, so that the multi-bump solution U is everywhere greater than δ_ν, for every integer ν. But the sequence δ_ν converges to 0, so that U is non-negative.

It remains to show that U is nowhere vanishing. But if M is an upper bound for the potential function G, we have

$$-\Delta U + MU = (M - G(x))U \geq 0$$

since $U \geq 0$. Using the fundamental solution E_M to the Helmholtz equation $-\Delta u + Mu = 0$, we find :

$$U(x) = \int_{\mathbb{R}^n} \big(M - G(y)\big)U(y)\,E_M(x-y)\,dy.$$

Since $E_M > 0$, it follows that if $U(x) = 0$ for one x, then U vanishes identically. This is of course impossible, since U is close to the function $\sum_{b\in B_0} u_\varepsilon^b$ which achieves local maxima near b/ε. Hence the function U is strictly positive.

$\square$

7 A worked example

The existence, under some natural assumptions that allow non-periodic V, of single bump solutions having the regularity, uniformity and decay properties assumed in section 2, will be the subject of a succeeding paper. In order to give an example here we have recourse to the case of periodic V, where uniformity comes free, and use one dimension, where the ground state with the required properties is explicitly known. The following problem was already treated by Thandi [18]:

$$-u'' + V(\varepsilon x)u - u^3 = 0, \quad x \in \mathbb{R} \tag{NLS}$$

where V is a C^∞ periodic function such that $V(x) > h > 0$.

Let us proceed to check the assumptions one by one. In this setting, $F(a,u) = au - u^3$ is obviously of class C^∞. The interval I is $]0,\infty[$ as we shall shortly see. The positivity assumption (P) then holds.

The ground state. Let $a \in I$. The ground state solution to

$$-u'' + au - u^3 = 0$$

can be explicitly constructed. It is namely

$$\phi_a(x) = \sqrt{a}\ \mathrm{sech}(\sqrt{a}x).$$

Assumptions (Φ1), (Φ2), (Φ4) and (Φ5) are readily checked. Assumption (Φ3) follows from observing that the equation $-v'' + av - 3\phi_a^2 v = 0$ already has the solution ϕ_a', which decays with its derivatives at $\pm\infty$, so that the Wronskian of ϕ_a' and a second solution $v \in C^{2,\lambda}$ must decay at $\pm\infty$. But for this equation the Wronskian is a constant so that v cannot be independent of ϕ_a'. Moreover the operator $-\frac{d^2}{dx^2} + a - 3\phi_a^2 : C^{2,\lambda} \to C^{\lambda}$ is a Fredholm operator of index 0, so its range has codimension 1.

The set B and the single-bump solutions. From now on b will be a fixed non-degenerate critical point of V and $a = V(b)$. We let B be the complete set of translates $b + \gamma$ by the periods of V. In this case $V(b + \gamma) = V(b) = a$ so that V and its derivatives are constant on B and assumptions (B1) and (B2) are trivial. Notwithstanding the simple form of this problem the existence and structure of the single-bump solution is scarcely an easy problem.

The asymptotic form

$$u_\varepsilon^b(x) = \phi_a\left(x - \frac{b}{\varepsilon} + s_\varepsilon^b\right) + \varepsilon^2 w_\varepsilon^b\left(x - \frac{b}{\varepsilon} + s_\varepsilon^b\right)$$

for the single bump solution corresponding to the critical point b was given in [8]. It is known by [8] that the function w_ε^b, which is orthogonal to ϕ_a', is in H^2 and tends to η^b in H^2. This function is the unique solution $v = \eta$ in H^2 to

$$-v'' + av - 3\phi_a^2 v = -\frac{1}{2}\phi_a(x)V''(a)x^2$$

orthogonal to ϕ_a'. The other solutions $u_\varepsilon^{b+\gamma}$, given by [8], are just the translates $u_\varepsilon^{b+\gamma}(x) = u_\varepsilon^b\left(x - \frac{\gamma}{\varepsilon}\right)$ and similarly $w_\varepsilon^{b+\gamma}(x) = w_\varepsilon^b\left(x - \frac{\gamma}{\varepsilon}\right)$ and $s_\varepsilon^{b+\gamma} = s_\varepsilon^b$. This follows from the uniqueness inherent in the implicit function theorem.

Thus uniformity with respect to γ comes free and we only have to check regularity and decay (with uniformity with respect to ε) for the one point b. Note first that $H^2(\mathbb{R})$ is embedded into $C^1(\mathbb{R})$ and $u_\varepsilon^b \in H^2(\mathbb{R})$. Since V has bounded derivatives to all orders we find by bootstrapping that $u_\varepsilon^b \in C^k(\mathbb{R})$ for all k. Moreover the bound on $||u_\varepsilon^b||_{H^2(\mathbb{R})}$ uniform with respect to ε transfers to a bound on $||u_\varepsilon^b||_{C^k(\mathbb{R})}$ uniform with respect to ε.

The same embedding of H^2 into C^1 gives that w_ε^b is a bounded family in $C^1(\mathbb{R})$ and that $w_\varepsilon^b \to \eta^b$ in $C^1(\mathbb{R})$. Bootstrapping the equation satisfied by η^b shows that it too belongs to $C^k(\mathbb{R})$ for all k. This is enough to give all the regularity assumptions in (U1), (U2) and (U3).

Uniform exponential decay. Assumption (U4) is not trivial even for this simple example. Recall that b is a fixed critical point.

We first show that the function $v_\varepsilon := \varepsilon^2 w_\varepsilon^b$ decays at infinity uniformly with respect to ε. Let us prove this by contradiction. We assume that there exists $\delta > 0$, a sequence $0 < \varepsilon_i < \varepsilon_0$ and real numbers x_i with $|x_i| > i$ such that $|v_{\varepsilon_i}(x_i)| > \delta$ for every i. It follows from (U3) that v_ε is Lipschitz continuous, and the constant can be chosen independently of ε. Hence $|v_{\varepsilon_i}(x)| > \delta/2$ whenever $|x - x_i| < t$, where t does not depend on i. Integrating this inequality over the interval $[x_i - t, x_i + t]$ gives :

$$\int_{x_i-t}^{x_i+t} |v_{\varepsilon_i}(x)|^2 \, dx \geq C$$

where C is a constant independent of i. If a subsequence of ε_i converges to 0, this contradicts the known fact that v_ε converges to 0 in L^2. We may therefore assume that ε_i converges to a non-zero $\widetilde{\varepsilon}$. But then we obtain

$$\int_{x_i-t}^{x_i+t} |v_{\widetilde{\varepsilon}}(x)|^2 \, dx \geq C/2$$

for sufficiently large i, contradicting the fact that $v_{\widetilde{\varepsilon}} \in L^2$, since $|x_i| \to +\infty$. Here we used another fact from [8]: the map $\varepsilon \to w_\varepsilon^b$ is continuous to the L^2-topology. In fact the preceding argument boils down to a simple fact: a subset of $L^2(\mathbb{R})$ that is relatively compact and uniformly equicontinuous decays uniformly at infinity.

We have seen that the function $u_\varepsilon^b(\cdot + b/\varepsilon - s_\varepsilon^b)$ has uniform decay at infinity. To finish the proof, we notice that this function satisfies the linear equation

$$-v'' + G_\varepsilon(x)v = 0$$

where

$$G_\varepsilon(x) = V\big(b + \varepsilon(x - s_\varepsilon^b)\big) - u_\varepsilon^b(x + \xi_\varepsilon^b)^2$$

By the assumption $V(x) > h$ and the uniform decay obtained above we see that, for every $\delta > 0$, there is a compact set K of $\mathbb{R}^n$, independent of ε, such that $G_\varepsilon(x) > h - \delta$ for every x not in K.

We may now appeal to general decay properties of eigenfunctions, for example [16, theorem 3.19], to obtain that, if $\mu < \sqrt{h}$,

$$\Big| u_\varepsilon^b\big(x + \xi_\varepsilon^b\big) \Big| \leq C \|u_\varepsilon^b\|_{L^\infty} \, e^{-\mu|x|}$$

for every $x \in \mathbb{R}^n$. The constant C does not depend upon ε; it only depends on R chosen such that $G_\varepsilon(x) > \mu^2$ for $|x| > R$ and we saw above that such a quantity can be found independent of ε. As for $\|u_\varepsilon^b\|_{L^\infty}$ we know that u_ε^b is a bounded family in $C(\mathbb{R})$.

The exponential decay of $D^\alpha u_\varepsilon^b(x + \xi_\varepsilon^b)$ follows by interpolation since we know that all derivatives of u_ε^b are bounded.

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